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Existence of Monotone Equilibria in First-Price Auctions with Resale*

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Abstract

Existence of a monotone pure-strategy perfect Bayesian equilibrium is proved for a multistage game of first-price auctions with interbidder resale, with any reserve price and any finite number of ex ante different bidders. Endogenous gains at resale complicate the winner's curse and upset previous fixed-point methods to prove existence of monotone equilibria. This paper restructures the fixed-point approach with respect to previously unknown comparative statics of the resale mechanisms strategically chosen after the auction. Despite speculation possibilities and the discontinuity-inducing uniform tie-breaking rule, at our equilibrium any above-reserve bid that stands a chance to win is strictly increasing in the bidder's use value.

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1 Introduction

Analyses of economic institutions are based on existence of equilibria of the underlying games. Among them first-price auctions, widely used in practice, are of particular theoretical interest because a discontinuity problem, caused by tying bids, may upset standard arguments of equilibrium existence. To solve this discontinuity problem sophisticated methods based on fixed-point theorems have been developed, one guaranteeing existence of monotone pure-strategy equilibria due to Athey [1], McAdams [10], Reny and Zamir [16], and Reny [15], and the other for mixed-strategy equilibria, augmented with endogenous tie-breaking rules, due to Jackson, Simon, Swinkels and Zame [5].[1] However, neither method has been applied to dynamic games such as auctions with resale.[2] With resale, foundational assumptions need to be reexamined with respect to the continuation play at resale. For example, a main hurdle for the fixed-point approach to monotone equilibria is the winner's curse, which has been handled in the literature by bounding it with sufficiently strong primitive assumptions. But resale would endogenize the winner's curse and renders it unbounded a priori, as a bidder could magnify the winner's curse for the rivals by acting as a high-bidding speculator so that his rivals might want to lose now and buy the good at resale. This paper restructures the monotone pure-strategy fixed-point approach with respect to comparative statics of resale thereby proving existence of a perfect Bayesian equilibrium, with increasing bid functions, for a two-stage game of any first-price sealed-bid auction with resale.

1.1 Where Is the Monotone Fixed-Point Approach Stuck?

The general idea of this fixed-point approach, dating back to the general equilibrium literature,[3] is to approximate the original economy by some sequence of finite economies where equilibria exist and then prove that a limit point of the sequence of such *approximation equilibria* is an equilibrium of the original one. For auctions, the main impediment to such passing-to-limit arguments is a discontinuity problem caused by ties. For instance, in Figure 1, each bidder $i \in \{1, 2, 3\}$ plays an equilibrium bidding strategy β_i^m, a nondecreasing

[1] Kotowski [6] has a recent application of the fixed-point methods in auctions with budget constraints.

[2] The conceptual awkwardness of the no-resale assumption has been noted by Zheng [19] and Hafalir and Krishna [4]. The possibility of resource misallocation, which may occur at equilibrium in first-price auctions among ex ante different bidders given the no-resale assumption, induces bidders to attempt resale.

[3] For example, Werner [18] and Magill and Quinzii [9].

function from his type t_i to a bid, in the approximation auction game indexed by m; when the sequence $(\beta_1^m, \beta_2^m, \beta_3^m)_{m=1}^{\infty}$ converges to its limit, a nonvanishing mass of bids, submitted by bidder 1 of types in $[a_1, z_1]$ and bidder 2 of types in $[a_2, z_2]$, are clustered within an interval collapsing into the point x, which becomes an *atom* (while bidder 3's types that bid within the cluster vanish into a point z_3). The crucial stage of the fixed-point approach is to demonstrate a contradiction to the approximation equilibria by arguing that some types of at least one of the bidders, say some elements in $[a_2, z_2]$, strictly prefer to deviate from their β_2^m-bids within the cluster at x to a bid say x' slightly above the cluster. This no-tie argument, due

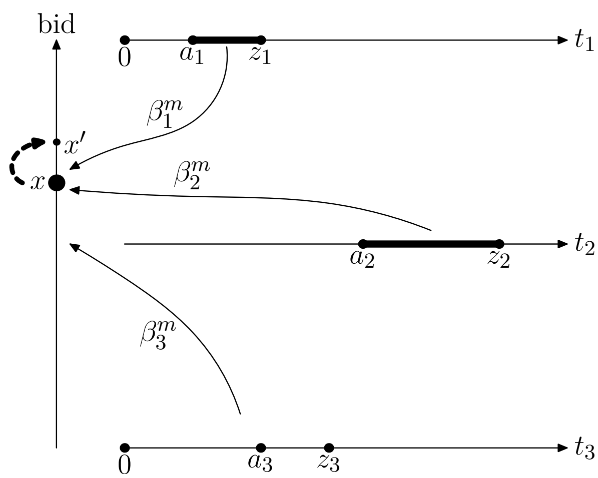

Figure 1: A tying situation

to Athey [1] and now standard within the fixed-point literature, can be summarized into two steps (Claims 1 and 2, Athey [1, Appendix]), illustrated here from bidder 2's viewpoint:

i. One needs to prove that, as bidder 2's type increases from a_2 to z_2, his preference to winning increases strictly, and eventually, with sufficiently high types, he strictly prefers to win conditional on the *winning event* that he can win with the β_2^m-bids within the cluster at x, which roughly corresponds to the event "$(t_1, t_3) \in [0, a_1] \times [0, z_3]$".

ii. For the desired contradiction it suffices to show that the types obtained in the previous step strictly prefer to deviate to x' from their β_2^m-bids within the cluster at x. This was done by proving that their expected net gains from winning cannot decrease when they consider only the event in which the deviation is pivotal, i.e., that bidder 2 cannot prefer less to win when the conditioned event moves from the winning event "$(t_1, t_3) \in [0, a_1] \times [0, z_3]$" up to the *pivotal event* "$(t_1, t_3) \in [a_1, z_1] \times [0, z_3]$".

To see the troubles, consider an independent private values model where t_i is bidder i's use value of the good for sale. Step (i) can fail because a bidder with high types, say the elements of $[a_2, z_2]$ in Figure 1, may eventually acquire and consume the good whether he wins it now or buys it later at resale. Then the type t_2 in bidder 2's payoff as a winner is canceled out by the t_2 in his payoff as a loser, so his net gain from winning does not increase in t_2, and $[a_2, z_2]$ need not contain a type that strictly prefers to win, contrary to Step (i).

To consider a case where Step (ii) is unsalvageable, suppose that, within the bid cluster at x in Figure 1, bidder 1's bids are above bidder 2's, so that bidder 1 wins when they both bid in the cluster. Thus, the winning event for bidder 2, when he bids within the cluster, is "$(t_1, t_3) \in [0, a_1] \times [0, z_3]$", while the pivotal event for bidder 2's deviation from the cluster to x' is "$(t_1, t_3) \in [a_1, z_1] \times [0, z_3]$". If Athey's Step (ii) worked, bidder 2's preference to win would not diminish when the conditioned event moves from the winning event to the pivotal one. Given resale, however, the opposite can be true. For instance, let the probability of $[a_3, z_3]$ be so large that, conditional on the winning event $[0, a_1] \times [0, z_3]$, if bidder 2 loses then with a large probability he buys the good from the types $[a_3, z_3]$ of bidder 3. By contrast, conditional on the pivotal event $[a_1, z_1] \times [0, z_3]$, if bidder 2 loses, he buys the good from bidder 1 with types in $[a_1, z_1]$. Since $[a_3, z_3]$ is higher than $[a_1, z_1]$ in strong-set order, the resale price offered to bidder 2, in expectation, is higher in the winning event than in the pivotal event. Thus, when the conditioned event moves up to the pivotal one, bidder 2's expected payoff from losing, or roughly speaking the *winner's curse*, becomes higher. On the other hand, bidder 2's payoff from winning is invariant to his rivals' types because, from Figure 1, $a_2 > z_3 > z_1$ and hence if he wins then he will consume the good to obtain its use value t_2. Consequently, when he takes into account that his deviation is pivotal, bidder 2 prefers strictly less to win, contrary to Step (ii).

The fundamental reason why Athey's no-tie argument does not work here is that a monotonicity assumption in the literature may fail given resale. The assumption stipulates that a bidder's ex post net payoff from winning is nondecreasing in his rivals' types (e.g., A.1.iii of Reny and Zamir). With resale, by contrast, a winner's payoff may fail to be nondecreasing in his rivals' types because the optimal resale mechanism may resell the good to a subsidized bidder who pays a lower price than someone else, so the ex post resale revenue may decrease when a subsidized bidder's type rises to become the buyer at resale. A loser's payoff may fail to be nonincreasing because a loser's gain from trading with reseller j may be larger than that with reseller k. Thus, when j has a slightly higher type to become the reseller instead of k, this bidder's ex post payoff increases. Hence the ex post net gain from winning may fail to be nondecreasing in the rivals' types.

In addition to the no-tie argument, two other important conditions, which did not appear difficult in the received literature, become problematic given resale. One is single crossing, crucial to guarantee existence of the aforementioned approximation equilibria. The other is payoff security, indispensable to deliver the passing-to-limit result in the literature. The *single-crossing* condition says that if a bidder prefers a high bid to a low one then the preference remains so when his type gets higher. The primitive assumption from which the literature obtained this condition is single crossing for every possible realized type profile (e.g., A.1.iv of Reny and Zamir). With resale, the assumption fails when an increase of a bidder's type turns him from a speculator to a consumer, with high types of his rivals.[4] The *payoff-security* condition says that bidding slightly above an atom of the rivals' bids does not make a bidder worse-off than bidding at the atom. In the literature, verification of this condition is simply Step (ii),[5] which as illustrated above can fail with resale.

1.2 A Road Map of This Paper

This paper is devoted to overcoming the challenges that resale presents to the fixed-point approach. To capture the endogenous nature of resale, we assume that the resale mechanism is a reseller-optimal auction à la Myerson [13] based on post-auction beliefs. Athey's critical steps are restructured with respect to previously unknown comparative statics properties of the Myerson resale auction, with initial bids or post-auction beliefs being the parameters.

Our journey starts with an increasing-difference theorem (Theorem 1), which through its single-crossing implication ensures existence of the aforementioned approximation equilibria. Then, to pass their equilibrium condition to the limit, two building blocks are established. First is a no-tie theorem saying that ties do not occur at a limit point of a sequence of such approximating equilibria (Theorem 2). The second is a payoff-security theorem saying

[4] While the higher bid brings about higher revenues for the speculator-type since he charges higher resale prices due to the higher posterior about the willingness-to-pay of his clientele, the consumer-type, who benefits from none of such revenue effect, strictly prefers the lower bid, which costs him less. This also upsets a slightly weaker single-crossing assumption proposed by Quah and Strulovici [14, Th. 4(c), p28].

[5] For example, the displayed formula (A.5) in Reny and Zamir [16].

that a bidder suffers little loss from avoiding his rivals' atoms at this limit point (Theorem 3). Theorems 2 and 3 together deliver the existence theorem (Theorem 4).

The first step of our no-tie argument is to prove that, if a tie at the limit occurs then there exists a *dominant* bidder whose probability of winning the tie converges to one (Lemma 8). In Figure 1, for instance, the infimum a_1 of bidder 1's types that bid within the cluster at x is less than all elements of $[a_2, z_2]$, bidder 2's types bidding in the cluster. Consequently, with types being use values of the good, conditional on the pivotal event "$(t_2, t_3) \in [a_2, z_2] \times [0, z_3]$" of the bid increase from the cluster to x', bidder 1 would gain nothing from trading with the reseller player 2. I.e., the the bid increase renders zero winner's curse for bidder 1. On the other hand, the bid increase generates a *revenue effect* by adding a mass of high types $[a_2, z_2]$ to bidder 1's clientele thereby generating a significant increase in his expected resale revenue (Lemma 9, due to a property of the optimal resale mechanism proved in §A.1.2). Thus, bidder 1 with types nearby a_1 would strictly prefer to deviate unless within the cluster his bids are almost exclusively on the top layer so that he mostly outbids the tying rivals.[6] Hence bidder 1 is the dominant bidder.

To derive a contradiction from the supposed occurrence of a tie, our next step is to prove that some bidder who is supposed to bid just below the dominant rival within the tying cluster, such as bidder 2 in Figure 1, strictly prefers to deviate to a bid slightly above the cluster. The proof, from §5.2.1 to §5.2.5, is nontrivial because the winner's curse for bidder 2 is not negligible. Contrary to the case of bidder 1, even the infimum a_2 of the atom-bidding types of bidder 2 can gain from buying the good at resale from some atom-bidding types of bidder 1, as $a_2 > a_1$. This nontrivial winner's curse is handled in two substeps. First, we prove that if $[a_2, z_2]$ contains some sufficiently high types then for such types of bidder 2 the winner's curse is more than outweighed by the "winner's blessing" (payoff from winning conditional on the pivotal event). Then he strictly prefers the deviation (§5.2.3, due to a property of the optimal resale mechanism proved in §A.1.1). Second, in the other case, we find some types in $[a_2, z_2]$ for whom the winner's curse is nearly balanced by the winner's

[6] Although the winner's curse is null in this case, Athey's no-tie argument, without taking into account the revenue effect, still cannot be replicated to prove that bidder 1 strictly prefers the higher bid. Even if her Step (i) works, so that bidder 1's preference to win strictly increases in his type on $[a_1, z_1]$ conditional on his winning event, his preference may still be reversed when the conditioned event switches to the *pivotal* event. That is because his ex post payoff from winning may fail to be nondecreasing in his rivals' types, as explained above regarding the monotonicity assumption.

blessing. This is done by deducing the viability of bidder 2's deviation from the profitability of bidder 1's on-path action despite information asymmetry between them (§5.2.4).[7] Then the revenue effect of the deviation, as in the case for bidder 1 in the previous paragraph, implies bidder 2's strict incentive to deviate (§5.2.5), which delivers the no-tie theorem.

In the received literature, a no-tie theorem would have sufficed the passing-to-limit argument, as the aforementioned payoff-security condition was obtained by repeating Step (ii) in Athey's argument. Not so with resale, because as explained previously the monotonicity assumption may fail. With this assumption, Athey's Step (ii) is accomplished without relying on any equilibrium condition. Without the assumption, our no-tie argument relies on the condition that the deviant bidder is supposed to bid at the cluster according to the approximation equilibria (so that the deviation costs him little increase of payment). But such an equilibrium condition is unavailable when payoff security is being considered.

Our solution, Theorem 3, is due to a general feature of the increasing-difference theorem that allows bids to be atoms. This general feature is obtained by comparing resale mechanisms in resale environments that are irregular, necessitating the ironing procedure (Appendix C). Increasing difference allows us to establish payoff security for low types, for whom the winner's curse of skipping an atom is negligible as in the first step of our no-tie argument, and then extend payoff security to all higher types. Theorem 3 is also due to comparative statics of a reseller's expected revenues when the distribution of a bidder turns from an irregular to a regular one (Corollary 4). The complication of irregular resale settings is unavoidable to a reseller whose winning bid is an atom of his rivals (Lemma 2.b), and such an atomic case is unavoidable to us because payoff security is precisely about bidding at a rival's atom (whereas the no-tie theorem is about bidding nearby the atom). Both Theorems 2 and 3 combined, the existence proof is complete.

The existence theorem is more general than previous results in first-price auctions with resale in that it allows for any reserve price and any finite number of differently distributed bidders while the previous literature assumed either two bidders or at most two kinds of bidders ex ante, with bidders of the same kind drawn from the same distribution. Notwith-

[7] The deduction, consisting of Lemmas 11 and 12, is based on two nontrivial facts. First, bidder 2 can nearly mimic bidder 1's optimal resale mechanism in the event of the tie, largely due to the fact that bidder 1 is the dominant rival. Second, the expected revenue produced by a fixed Myerson auction does not decrease when the weight of a bidder's type is pushed upward (Lemma 19, proved here despite the fact that the ex post revenue generated by a Myerson auction need not be nondecreasing in a bidder's type).

standing some remarkable results in this literature, such as Garratt and Tröger [2] in mixed strategies and Hafalir and Krishna [4], Lebrun [7, 8] and Virág [17] in pure strategies, the two-distribution assumption has been crucial to their differential equations method.

Nevertheless, the existence theorem is restricted by assumptions such as undisclosed losing bids in the initial auction, common infimum for bidders' prior supports, and a reseller's power to choose resale mechanisms, though these are common assumptions in the current auction-resale literature such as those cited above as well as Zheng [19] and Garratt, Tröger and Zheng [3].[8] Now that the existence proof has shown it feasible to extend the fixed-point approach beyond its previous confines of treating an auction as an isolated event, investigations of its further expansion are at hand.

2 The Model

2.1 The Auction-Resale Game

There are two periods, a finite set I of bidders, and an indivisible good. For each $i \in I$, bidder i's *type*, or use value of the good, is independently drawn from a commonly known distribution F_i, with the realized value privately known to i. In period one, every bidder i submits as his bid an element of $\{l\} \cup B_i$, where the *losing bid* $l < 0$ amounts to nonparticipation in the period-one auction, and $B_i \subseteq [r, \infty)$ is the set of *serious bids* admissible for bidder i, with reserve price $r \geq 0$ for all bidders. Ties are broken randomly and uniformly with equal probabilities. If the good is sold then, after the winner is selected, the highest bid and the winner's identity are announced publicly, with nothing else disclosed,[9] and the winner pays for the good at the price equal to his winning bid. Then period two starts and the period-one winner chooses a selling mechanism that offers resale to the other bidders in I, called *losing bidders*. A selling mechanism is any game form to be played by the losing bidders. After the players have acted given this mechanism, the entire game ends.

Each bidder is risk-neutral in his payoff, equal to his use value, if and only if he is the

[8] Zheng [19] did not assume common infimum of the priors but made some other assumptions. Hafalir and Krishna [4] and Lebrun [7, 8] considered some other disclosure policies and weaker bargaining power of the reseller based on the two-distribution assumption and take-it-or-leave offers as the resale mechanism.

[9] If the action of a losing bidder is also disclosed, pure-strategy equilibrium is unlikely to exist unless the loser gets to choose the resale mechanism.

final owner of the good, plus the total money transfer from others to him over both periods.

Assume for any bidder i that the prior F_i has differentiable and strictly positive density f_i on its support $T_i := \left[0, \bar{t}_i\right]$, with prior virtual utility $t_i - (1 - F_i(t_i))/f_i(t_i)$ having strictly positive derivative with respect to t_i on T_i. Denote $T_{-i} := \Pi_{k \in I \setminus \{i\}} T_k$ and $T := \Pi_{k \in I} T_k$.[10]

A profile $(\beta_i)_{i \in I}$ of bid functions, with $\beta_i : T_i \to \{l\} \cup B_i$ for each $i \in I$, is said *monotone* if and only if β_i is a weakly increasing function for each $i \in I$, i.e., everyone's period-one bid is weakly increasing in his use value of the good.

2.2 Boldfaced Symbols for Random Variables

Denote bidder i's type by $\boldsymbol{t}_i$ as the random variable and t_i as the realized value. Denote $\boldsymbol{t}_{-i} := (\boldsymbol{t}_k)_{k \in I \setminus \{i\}}$ and $t_{-i} := (t_k)_{k \in I \setminus \{i\}}$ as the random vector and the realization for the type profile across rivals of i. Analogously, denote $\boldsymbol{t} := (\boldsymbol{t}_i, \boldsymbol{t}_{-i}) := (\boldsymbol{t}_k)_{k \in I}$, $t := (t_i, t_{-i}) := (t_k)_{k \in I}$, $\boldsymbol{t}_{-(i,j)} := (\boldsymbol{t}_k)_{k \in I \setminus \{i,j\}}$ and $t_{-(i,j)} := (t_k)_{k \in I \setminus \{i,j\}}$. Denote $\mathbb{E}[g(\boldsymbol{x})]$ for the expected value of any function g of the random variable or random vector $\boldsymbol{x}$, with the random variable/vector boldfaced, based on the prior distributions. Denote $\mathbb{E}[g(\boldsymbol{x}) \mid E]$ for the expected value conditional on event E, $\mathbf{1}[E]$ for the indicator function of event E, and $\Pr\{E\} := \mathbb{E}\left[\mathbf{1}[E]\right]$.

3 The Endogenous Payoff Functions

We shall derive a bidder's expected payoff in the auction-resale game from a continuation equilibrium at the resale stage, which implements a reseller-optimal auction à la Myerson [13].

3.1 Continuation Equilibrium at Resale

3.1.1 Atoms and Inverse Images of Bids

For any weakly increasing function $\beta_i : T_i \to \mathbb{R}$ and any $b \geq \beta_i(0)$, denote $\beta_i^{-1}(b)$ for the inverse image and, letting $\sup S := \inf S := 0$ when a subset S of T_i is empty, denote

$$\beta_{i,\inf}^{-1}(b) \quad := \quad \sup\{t_i \in T_i : \beta_i(t_i) < b\}, \tag{1}$$

$$\beta_{i,\sup}^{-1}(b) \quad := \quad \sup\{t_i \in T_i : \beta_i(t_i) \leq b\}. \tag{2}$$

[10] The common infimum assumption of bidders' prior supports is used in Lemmas 15 and 17. The positive-derivative assumption of prior virtual utilities is slightly stronger than the usual one that requires only strict monotonicity. The strengthening is needed in Corollary 2 and Lemmas 2.a.ii, 12 and 15.

Note that if $\beta_i^{-1}(b) \neq \varnothing$ then $\beta_{i,\inf}^{-1}(b) = \inf \beta_i^{-1}(b)$ and $\beta_{i,\sup}^{-1}(b) = \sup \beta_i^{-1}(b)$.

For any bidder i, an *atom of* β_i means a bid $b \in B_i$ such that $\beta_i^{-1}(b)$ is a nondegenerate interval, i.e., $\beta_{i,\inf}^{-1}(b) < \beta_{i,\sup}^{-1}(b)$. An *atom of* β_{-i}, with $\beta_{-i} := (\beta_j)_{j \neq i}$, means an atom of β_j for some $j \in I \setminus \{i\}$. Likewise, an atom of $\beta := (\beta_j)_{j \in I}$ means an atom of β_j for some $j \in I$.

3.1.2 Public Histories and Posterior Beliefs

If bidder i wins with bid b_i in period one (so $b_i > l$, i.e., $b_i \in B_i$) then (i, b_i) denotes the commonly known *public history*. Given any public history (i, b_i), with every losing bidder k ($k \neq i$) having played according to β_k, the posterior distribution $F_k(\cdot \mid i, b_i, \beta)$ of $\boldsymbol{t}_k$ is derived from Bayes's rule based on the observation that k has been defeated either because $\beta_k(\boldsymbol{t}_k) < b_i$ or because $\beta_k(\boldsymbol{t}_k) = b_i$ and k did not win the tie-breaking lottery.

Lemma 1 *For any public history (i, b_i), any monotone profile β, and any $k \neq i$, the density $f_k(\cdot \mid i, b_i, \beta)$ of $F_k(\cdot \mid i, b_i, \beta)$ is finite and strictly positive on its support $\left[0, \beta_{k,\sup}^{-1}(b_i)\right]$; if b_i is not an atom of β_k then $f_k(\cdot \mid i, b_i, \beta)$ is continuous on this* posterior *support; else $f_k(\cdot \mid i, b_i, \beta)$ is continuous at all but one point in the posterior support.*

Proof Appendix C. ■

3.1.3 Posterior Virtual Utilities

For each losing bidder $k \in I \setminus \{i\}$ in public history (i, b_i), define $V_{k,b_i,\beta} : T_k \to \mathbb{R}$ by

$$V_{k,b_i,\beta}(t_k) := V_k(t_k \mid b_i, \beta) := \begin{cases} t_k - \frac{1-F_k(t_k|i,b_i,\beta)}{f_k(t_k|i,b_i,\beta)} & \text{if } t_k \leq \beta_{k,\sup}^{-1}(b_i) \\ \beta_{k,\sup}^{-1}(b_i) & \text{if } t_k \geq \beta_{k,\sup}^{-1}(b_i), \end{cases} \tag{3}$$

and define the *posterior virtual utility* function for losing bidder $k \neq i$ to be either $V_{k,b_i,\beta}$ if b_i is not an atom of β_k, or the ironed version of $V_{k,b_i,\beta}$ according to Myerson's [13] procedure if b_i is an atom of β_k. By the previous and the next lemmas, $V_{k,b_i,\beta}$ fails to be monotone and hence ironing is needed precisely when the winning bid b_i is an atom of β_k. Denote k's posterior virtual utility by $\overline{V}_{k,i,b_i,\beta}(t_k)$ or $\overline{V}_k(t_k \mid i, b_i, \beta)$.[11]

Lemma 2 *There exists $\lambda > 0$ such that, for any public history (i, b_i), any monotone profile β, and any $k \neq i$:*

[11] When the winning bid b_i is an atom of β_k, the posterior distribution of $\boldsymbol{t}_k$ depends on i by Eq. (68). Hence the notation i for the winner in the ironed posterior virtual utility function $\overline{V}_{k,i,b_i,\beta}$ cannot be dropped.

a. if b_i is not an atom of β_k, then:

i. for any $t_k \in T_k$, $\overline{V}_{k,i,b_i,\beta}(t_k) = V_{k,b_i,\beta}(t_k)$ and, if $t_i \in \left[0, \beta^{-1}_{k,\sup}(b_i)\right]$,

$$V_{k,b_i,\beta}(t_k) = t_k - \frac{F_k\left(\beta^{-1}_{k,\sup}(b_i)\right) - F_k(t_k)}{f_k(t_k)}; \tag{4}$$

ii. $V_{k,b_i,\beta}$ is strictly increasing on $\left[0, \beta^{-1}_{k,\sup}(b_i)\right]$, at a rate greater than or equal to λ, and is constant on $\left[\beta^{-1}_{k,\sup}(b_i), \bar{t}_k\right]$;

iii. $\overline{V}_{k,i,b_i,\beta}$ is continuous on T_k;

b. for any $i, i' \in I \setminus \{k\}$, if $b' > b \geq r$, then $\overline{V}_{k,i,b,\beta} \geq \overline{V}_{k,i',b',\beta}$ on $\left[0, \beta^{-1}_{k,\sup}(b)\right]$.

Proof Appendix C. ■

3.1.4 Resale Mechanisms

Given any public history (i, b_i), by Lemma 1, Myerson's [13] characterization of optimal auctions is applicable to our reseller i's auction-design problem.[12] Specifically, the mechanism $M_i(b_i, t_i, \beta)$ defined below is optimal for the bidder-turned reseller i with type $t_i \in T_i$:

a. each losing bidder $k \neq i$ independently submits a report, say t_k, of his type;

b. for any $t_{-i} \in T_{-i}$, the good is allocated to

i. either a $k \neq i$ such that $\overline{V}_k(t_k \mid i, b_i, \beta) \geq \max\left\{t_i, \max_{j \notin \{i,k\}} \overline{V}_j(t_j \mid i, b_i, \beta)\right\}$

ii. or i when $t_i \geq \max_{j \neq i} \overline{V}_j(t_j \mid i, b_i, \beta)$;

if there are multiple k in alternative (i) or the inequalities in both (i) and (ii) are true, the tie is broken randomly and uniformly with equal probabilities;

c. if $k \neq i$ is allocated the good then the payment k delivers to i equals

$$p_{k,i,b_i,\beta}(t_{-k}) := \inf\left\{t'_k \in T_k : \overline{V}_{k,i,b_i,\beta}(t'_k) \geq \max\left\{t_i, \max_{j \in I \setminus \{i,k\}} \overline{V}_{j,i,b_i,\beta}(t_j)\right\}\right\}; \tag{5}$$

if k is not allocated the good then k pays zero to i.

[12] Myerson [13] assumed continuous density throughout a bidder's support while our posterior density may be discontinuous at one point (Lemma 1). But this difference does not affect Myerson's result. Also see Footnote 9 of Garratt, Tröger and Zheng [3] for an explanation why Myerson's result is applicable here despite the possibility that the reseller may be privately informed of her type.

Following directly from Myerson's result, we have—

Lemma 3 *For any public history (i, b_i), any $t_i \in T_i$ and any monotone profile β, if the posterior belief of $\boldsymbol{t}_j$ is $F_j(\cdot \mid i, b_i, \beta)$ for each $j \neq i$, then it is a continuation equilibrium for player i to choose $M_i(b_i, t_i, \beta)$ and everyone else to participate and be truthful.*

For any public history (i, b_i), if b_i is not an atom of β_{-i}, then Lemma 2.a implies that, for any losing bidder $k \neq i$, the posterior virtual utility function $\overline{V}_{k,i,b_i,\beta}$ is equal to the strictly increasing function $V_{k,b_i,\beta}$ on the posterior support $\left[0, \beta^{-1}_{k,\mathrm{sup}}(b_i)\right]$ of $\boldsymbol{t}_k$, hence for any t_{-k} such that bidder k of type t_k wins in $M_i(b_i, t_i, \beta)$ (i.e., $\max\left\{t_i, \max_{j \in I\setminus\{i,k\}} V_{j,b_i,\beta}(t_j)\right\} \leq \beta^{-1}_{k,\mathrm{sup}}(b_i)$), Eq. (5) is simplified to, with $V^{-1}_{k,b_i,\beta}$ denoting the inverse function of $V_{k,b_i,\beta}$,

$$p_{k,i,b_i,\beta}(t_{-k}) = V^{-1}_{k,b_i,\beta}\left(\max\left\{t_i, \max_{j \in I\setminus\{i,k\}} V_{j,b_i,\beta}(t_j)\right\}\right). \tag{6}$$

3.2 The Payoff from the Auction

3.2.1 The Indicator Function for Winning

The uniform tie-breaking rule corresponds to a random vector $(\boldsymbol{\rho}_i)_{i \in I}$ subject to two conditions: (i) for any realization $(\rho_i)_{i \in I}$, $\rho_i \in \{1, \ldots, |I|\}$ for any $i \in I$, and $\rho_i \neq \rho_j$ for any $i \neq j$; and (ii) any such realization has the same probability. The interpretation is that if $\rho_i > \rho_j$ then bidder i beats j in the coin toss when their bids are tied.

For any realization $(\rho_k)_{k \in I}$ of the uniform tie-breaking lottery, any $i \in I$, any $J \subseteq I\setminus\{i\}$, and any profile $(b_k)_{k \in J \cup \{i\}}$ of bids across bidders in $J \cup \{i\}$, write

$$(i, b_i) \succ_{(\rho_k)_{k \in I}} (b_k)_{k \in J}, \quad \text{or briefly} \quad (i, b_i) \succ (b_k)_{k \in J},$$

if and only if

$$b_i \in B_i \text{ and } \left[b_i > \max_{k \in J} b_k \text{ or } \left[b_i = \max_{k \in J} b_k \text{ and } \forall k \in \arg\max_{j \in J} b_j : \rho_i > \rho_k\right]\right].$$

And write $(i, b_i) \not\succ (b_k)_{k \in J}$ if and only if $(i, b_i) \succ (b_k)_{k \in J}$ is not true.

For example, $\mathbf{1}\left[(i, b_i) \succ (\beta_k(t_k))_{k \in I\setminus\{i\}}\right]$ is the indicator function for the event that bidder i wins, possibly after tie-breaking, with bids b_i from i and $\beta_k(t_k)$ from each rival k.

3.2.2 Ex Post Payoff for a Winner

For any public history (i, b_i) and any $(t_i, t_{-i}) \in T_i \times T_{-i}$, define $W_i(t_{-i} \mid b_i, t_i, \beta)$ to be the payoff for player i when i wins at the initial auction with bid b_i and offers resale via the Myerson auction $M_i(b_i, t_i, \beta)$ according to the continuation equilibrium specified in Lemma 3, when rivals of i abide by the monotone profile β_{-i} in period one and the profile of realized types across other players happens to be t_{-i}. That bidder i wins with bid b_i implies $b_i \in B_i$. For the case $b_i \notin B_i$, i.e., $b_i = l$, define $W_i(t_{-i} \mid l, t_i, \beta) := 0$.

If a serious bid b_i (i.e., $b_i \in B_i$) is not an atom of β_{-i}, one can derive from Lemmas 2 and 3 that, for all $t_{-i} \in \prod_{k \neq i} \left[0, \beta_{k,\sup}^{-1}(b_i)\right]$ except a set of measure zero and for any $t_i \in T_i$,

$$\begin{aligned} W_i(t_{-i} \mid b_i, t_i, \beta) \quad &= \quad t_i \mathbf{1}\left[t_i > \max_{k \neq i} V_k(t_k \mid b_i, \beta)\right] \\ &\quad + \sum_{j \neq i} p_{j,i,b_i,\beta}(t_{-j}) \mathbf{1}\left[V_j(t_j \mid b_i, \beta) > \max\left\{t_i, \max_{k \notin \{i,j\}} V_k(t_k \mid b_i, \beta)\right\}\right]. \end{aligned} \tag{7}$$

3.2.3 Ex Post Payoff for a Losing Bidder

For any distinct bidders $i \neq j$ and any $(t_i, t_{-i}) = (t_i, t_j, t_{-(i,j)}) \in T_i \times T_j \times T_{-(i,j)}$ such that $\beta_j(t_j) \in B_j$ and $\beta_j(t_j) \geq \beta_k(t_k)$ for all $k \in I \setminus \{i, j\}$, define $L_{ij}(t_{-i} \mid t_i, \beta)$ to be the payoff for player i when bidder j wins at the initial auction with bid $\beta_j(t_j)$ and offers resale via mechanism $M_j(\beta_j(t_j), t_j, \beta)$ according to the continuation equilibrium, when everyone is supposed by other players to abide by the monotone profile β in period one and the profile of realized types across bidders happens to be $(t_i, t_j, t_{-(i,j)})$.

Note that $L_{ij}(t_{-i} \mid t_i, \beta)$ is invariant to i's period-one bid b_i, due to the fact that reseller j in choosing resale mechanisms does not know the bids from the losing bidders.

If $\beta_j(t_j)$ is not an atom of β_{-j} then, as in the previous case for W_i, for any $i \neq j$, for all $t_{-(i,j)} \in \prod_{k \notin \{i,j\}} \left[0, \beta_{k,\sup}^{-1}(\beta_j(t_j))\right]$ but a set of measure zero, and for any $t_i \in T_i$,

$$L_{ij}(t_{-i} \mid t_i, \beta) = \left(t_i - p_{i,j,\beta_j(t_j),\beta}(t_{-i})\right) \mathbf{1}\left[V_{i,\beta_j(t_j),\beta}(t_i) > \max\left\{t_j, \max_{k \notin \{i,j\}} V_{k,\beta_j(t_j),\beta}(t_i)\right\}\right]. \tag{8}$$

Before the auction outcome is announced in period one, bidder i does not know who is the winner, but he knows that, at any realized type profile $t \in T$, if he loses the auction then the winner is selected from $I \setminus \{i\}$ with each rival $k \in I \setminus \{i\}$ bidding $\beta_k(t_k)$. Thus, i's ex post payoff from losing, given any realized type profile $(t_i, t_{-i}) \in T$, is equal to

$$L_i(t_{-i} \mid t_i, \beta) := \sum_{j \neq i} \Pr\left\{(j, \beta_j(t_j)) \succ (\beta_k(t_k))_{k \in I \setminus \{i,j\}}\right\} L_{ij}(t_{-i} \mid t_i, \beta). \tag{9}$$

3.2.4 Interim Expected Payoff

Denote $U_i(b_i, t_i, \beta)$ for type-t_i bidder i's expected payoff in the entire game from bidding b_i in period one followed by the continuation equilibrium specified by Lemma 3, provided that everyone else abides by the monotone profile β at period one. Thus,

$$\begin{aligned} U_i(b_i, t_i, \beta) &= \mathbb{E}\left[\mathbf{1}\left[(i, b_i) \succ (\beta_k(\boldsymbol{t}_k))_{k \in I \setminus \{i\}}\right](W_i(\boldsymbol{t}_{-i} \mid b_i, t_i, \beta) - b_i - L_i(\boldsymbol{t}_{-i} \mid t_i, \beta))\right] \\ &\quad + \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid t_i, \beta)\right], \end{aligned} \tag{10}$$

where the boldfaced letters inside the expectation operator $\mathbb{E}$ denote the random variables.

Since W_i and L_i are derived from the continuation equilibrium at resale, we obtain a perfect Bayesian equilibrium if the period-one bid functions best reply one another:[13]

Lemma 4 *If a monotone profile $(\beta_i)_{i \in I}$ of period-one bid functions constitutes a Nash equilibrium, across almost all bidder-types, with respect to the interim expected payoff functions $(U_i(\cdot, \cdot, \beta))_{i \in I}$ given by Eq. (10), then $(\beta_i)_{i \in I}$ coupled with the continuation play characterized in Lemma 3 constitutes a perfect Bayesian equilibrium of the auction-resale game.*

4 Increasing Difference

Based on comparative statics of the continuation equilibrium, Theorem 1 says that a bidder's expected-payoff difference due to an increase in his period-one bid is nondecreasing in his type. It should be noted that this theorem allows the bids to be atoms. Hence the proof needs to consider resale environments whose posterior virtual utilities need to be ironed.

Theorem 1 (increasing difference) *For any bidder i, any monotone profile β of bid functions, and any $b''_i > b'_i$, $U_i(b''_i, t_i, \beta) - U_i(b'_i, t_i, \beta)$ is weakly increasing in t_i throughout T_i.*

This property is due to a relationship between period-one bids and the final allocation after resale (Propositions 1 and 2), which say that higher period-one bids imply higher probabilities of being the final owner of the good. This relationship implies the increasing difference property through the payoff-equivalence routine in mechanism design. With notations and lemmas introduced in §4.1–§4.3, the proof of the theorem is completed in §4.4.

[13] The only kind of off-path events relevant to our consideration is that a bidder loses the initial auction while he is expected to win at every possible state. The deviation that causes such an off-path event can be easily deterred by some off-path belief adopted by the reseller.

4.1 Final Allocations

For any bidder i, any monotone profile β of bid functions, and any $t := (t_k)_{k\in I} \in T$, define:

- $Q_i(b_i, t, \beta)$ to be the probability with which i is the final owner in the continuation equilibrium (Lemma 3) conditional on the public history (i, b_i), when $b_i \in B_i$ and the realized type profile is t (if $b_i \notin B_i$, i.e., $b_i = l$, then define $Q_i(b_i, t, \beta) := 0$);

- $q_{ij}(t, \beta)$ to be the probability with which i is the final owner in the continuation equilibrium (Lemma 3) conditional on the public history $(j, \beta_j(t_j))$, when $\beta_j(t_j) \in B_j$ and the realized type profile is t (if $\beta_j(t_j) = l$ then define $q_{ij}(t, \beta) := 0$);

- $q_i(t, \beta)$ to be the probability with which i is the final owner when some rival of i wins the period-one auction and offers resale according to the continuation equilibrium, i.e.,

$$q_i(t, \beta) = \sum_{j\neq i} \Pr\left\{(j, \beta_j(t_j)) \succ (\beta_k(t_k))_{k\in I\setminus\{i,j\}}\right\} q_{ij}(t, \beta). \tag{11}$$

4.2 The Envelope Condition

For any bidder i, define (with boldfaced letters denoting random variables):

$$\begin{aligned}
\overline{W}_i(b_i, t_i, \beta) &:= \mathbb{E}\left[W_i(\boldsymbol{t}_{-i} \mid b_i, t_i, \beta) \mid (i, b_i) \succ (\beta_j(\boldsymbol{t}_j))_{j\in I\setminus\{i\}}\right], &(12)\\
\overline{L}_i(b_i, t_i, \beta) &:= \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid t_i, \beta) \mid (i, b_i) \not\succ (\beta_j(\boldsymbol{t}_j))_{j\in I\setminus\{i\}}\right], &(13)\\
\overline{Q}_i(b_i, t, \beta) &:= \mathbb{E}\left[Q_i(b_i, t_i, \boldsymbol{t}_{-i}, \beta) \mid (i, b_i) \succ (\beta_j(\boldsymbol{t}_j))_{j\in I\setminus\{i\}}\right], &(14)\\
\overline{q}_i(b_i, t_i, \beta) &:= \mathbb{E}\left[q_i(t_i, \boldsymbol{t}_{-i}, \beta) \mid (i, b_i) \not\succ (\beta_j(\boldsymbol{t}_j))_{j\in I\setminus\{i\}}\right]. &(15)
\end{aligned}$$

The next lemma follows from the Milgrom-Segal envelope theorem [11].

Lemma 5 *For any $i \in I$, any $b_i \in B_i \cup \{l\}$, and any monotone profile β, the functions $\overline{W}_i(b_i, \cdot, \beta)$ and $\overline{L}_i(b_i, \cdot, \beta)$ are absolutely continuous and, for any $t_i \in T_i$,*

$$\begin{aligned}
\overline{W}_i(b_i, t_i, \beta) &= \overline{W}_i(b_i, 0, \beta) + \int_0^{t_i} \overline{Q}_i(b_i, \tau_i, \beta) d\tau_i, &(16)\\
\overline{L}_i(b_i, t_i, \beta) &= \int_0^{t_i} \overline{q}_i(b_i, \tau_i, \beta) d\tau_i. &(17)
\end{aligned}$$

Proof Appendix D. ■

4.3 Initial Bids and the Final Allocation

Propositions 1 and 2 are about ex post probabilities conditional on the profile of realized types across all bidders, not to be confused with expected probabilities.

Proposition 1 *For any $i \in I$ and any monotone profile β, if $b_i'' > b_i'$ then $Q_i(b_i'', t, \beta) \geq Q_i(b_i', t, \beta)$ for any $t_i \in T_i$ and any t_{-i} such that $\beta_k(t_k) \leq b'$ for all $k \neq i$.*

Proof Appendix D. ■

Propositions 1 says that if a bidder wins the initial auction then his probability of eventually keeping the good cannot be lower had he submitted any higher bid. The intuition is that a higher winning bid would make the winner think more highly about the losing bidders' willingness to pay and hence set higher reserve prices. Consequently, given the same realized types, his mechanism results in no resale with a higher probability.

Proposition 2 *For any bidders $i \neq j$ and any monotone profile, $Q_i(b_i, t, \beta) \geq q_i(t, \beta)$ for any $t_i \in T_i$ and any $t_{-i} \in T_{-i}$ such that $b_i \geq \max_{k \neq i} \beta_k(t_k)$.*[14]

Proof Appendix D. ■

Proposition 2 says that a bidder is more likely to become the final owner of the good when he is the reseller than when he is a potential buyer at resale. This is similar to an elementary economics fact that a monopolist who cannot perfectly discriminate its potential buyers would under-supply its goods. The monopolist at resale, our reseller would not resell the good without a price markup above her own use value, while potential buyers are willing to pay for it at any price not exceeding their use values.

4.4 Proof of Theorem 1

By Eqs. (10), (12) and (13),

$$U_i(b_i, t_i, \beta) = \mathbb{E}\left[\mathbf{1}\left[b_i \succ \boldsymbol{t}_{-i}\right]\right]\left(\overline{W}_i(b_i, t_i, \beta) - b_i\right) + \mathbb{E}\left[\mathbf{1}\left[b_i \not\succ \boldsymbol{t}_{-i}\right]\right]\overline{L}_i(b_i, t_i, \beta), \tag{18}$$

where $b_i \succ \boldsymbol{t}_{-i}$ is a shorthand for i's winning event $(i, b_i) \succ (\beta_k(\boldsymbol{t}_k))_{k \in I \setminus \{i\}}$, and $b_i \not\succ \boldsymbol{t}_{-i}$ its complement. For any $b_i'' > b_i'$, let $\Delta U_i(t_i) := U_i(b_i'', t_i, \beta) - U_i(b_i', t_i, \beta)$. By Eq. (18),

$$\begin{aligned} \Delta U_i(t_i) \quad = \quad & \mathbb{E}\left[\mathbf{1}\left[b_i'' \succ \boldsymbol{t}_{-i}\right]\right]\left(\overline{W}_i(b_i'', t_i, \beta) - b_i''\right) - \mathbb{E}\left[\mathbf{1}\left[b_i' \succ \boldsymbol{t}_{-i}\right]\right]\left(\overline{W}_i(b_i', t_i, \beta) - b_i'\right) \\ & + \mathbb{E}\left[\mathbf{1}\left[b_i'' \not\succ \boldsymbol{t}_{-i}\right]\right]\overline{L}_i(b_i'', t_i, \beta) - \mathbb{E}\left[\mathbf{1}\left[b_i' \not\succ \boldsymbol{t}_{-i}\right]\right]\overline{L}_i(b_i', t_i, \beta). \end{aligned}$$

[14] Proposition 2 extends Lemma 1 of Garratt, Tröger and Zheng [3] to the ex post perspective.

Differentiate this equation with respect to t_i and then plug into the right-hand side the envelope equations (16) and (17) and the equations (14) and (15) for $\overline{Q}$ and $\overline{q}$ to obtain

$$\begin{aligned}\frac{\partial}{\partial t_i}\Delta U_i(t_i) &= \mathbb{E}\left[\mathbf{1}\left[b_i'' \succ \boldsymbol{t}_{-i}\right] Q_i(b_i'', t_i, \boldsymbol{t}_{-i}, \beta) - \mathbf{1}\left[b_i' \succ \boldsymbol{t}_{-i}\right] Q_i(b_i', t_i, \boldsymbol{t}_{-i}, \beta)\right] \\ &\quad + \mathbb{E}\left[\mathbf{1}\left[b_i'' \not\succ \boldsymbol{t}_{-i}\right] q_i(t_i, \boldsymbol{t}_{-i}, \beta) - \mathbf{1}\left[b_i' \not\succ \boldsymbol{t}_{-i}\right] q_i(t_i, \boldsymbol{t}_{-i}, \beta)\right].\end{aligned}$$

The right-hand side, after rearrangements, with notation β suppressed, is equal to

$$\underbrace{\mathbb{E}\left[\mathbf{1}\left[b_i' \succ \boldsymbol{t}_{-i}\right]\left(Q_i(b_i'', t_i, \boldsymbol{t}_{-i}) - Q_i(b_i', t_i, \boldsymbol{t}_{-i})\right)\right]}_{=:X} + \underbrace{\mathbb{E}\left[\mathbf{1}\left[b_i' \not\succ \boldsymbol{t}_{-i}, b_i'' \succ \boldsymbol{t}_{-i}\right]\left(Q_i(b_i'', t_i, \boldsymbol{t}_{-i}) - q_i(t_i, \boldsymbol{t}_{-i})\right)\right]}_{=:Y}.$$

For any t_{-i} at which the indicator function inside the integral X is nonzero, $b_i' \geq \max_{j\neq i}\beta_j(t_j)$ and hence Proposition 1 applies; for any t_{-i} at which the indicator inside Y is nonzero, $b_i'' \geq \max_{j\neq i}\beta_j(\boldsymbol{t}_j)$ and hence Proposition 2 applies. Thus, both X and Y are nonnegative. Hence $\frac{\partial}{\partial t_i}\Delta U_i(t_i) \geq 0$ for any t_i interior to T_i. This, coupled with the fact that $\Delta U_i(t_i)$ is absolutely continuous in t_i (since U_i by Eq. (18) is a linear combination of $\overline{W}_i$ and $\overline{L}_i$, each absolutely continuous in t_i by Lemma 5), implies the monotonicity of ΔU_i. ■

5 Equilibria of the Approximation Games

Based on Theorem 1, if the bid spaces in the initial auction are replaced by some discrete spaces, a monotone equilibrium exists. To obtain equilibrium in the original game, we shall prove that the equilibrium property of such approximation equilibria is passed onto the limit when the discrete bid spaces converge to the original one. A critical step of the proof is to show that ties occur with zero probability at the limit (Theorem 2). As explained in the Introduction, our no-tie argument is significantly different from that in the literature.

5.1 The Approximation Games

For any $m = 1, 2, \ldots$, define an *m-approximation game* by replacing for any bidder i the space B_i of serious bids with a discrete set B_i^m such that

$$i \neq j \Longrightarrow B_i^m \cap B_j^m = \varnothing, \tag{19}$$

$$m < m' \Longrightarrow B_i^m \subseteq B_i^{m'}, \tag{20}$$

$$\min\left\{|b_i - b_i'| : b_i, b_i' \in B_i^m; b_i \neq b_i'\right\} = 2^{-m}, \tag{21}$$

$$\lim_{m\to\infty}\min B_i^m = r, \quad \lim_{m\to\infty}\sup B_i^m = \infty.$$

The main condition is Eq. (19), which ensures that, in any m-approximation game, a bidder's serious bid is never an atom of a rival's bid function.[15] Consequently,

$$(i, b_i) \succ (\beta_k(\boldsymbol{t}_k))_{k \in I \setminus \{i\}} \Longleftrightarrow b_i > \max_{j \neq i} \beta_j(\boldsymbol{t}_j), \tag{22}$$

hence a bidder's winning event is simplified. Another consequence is that the posterior virtual utility functions are simplified to Eq. (4) due to Lemma 2.a.i.

For any $m = 1, 2, \ldots,$, a profile $(\beta_i^m)_{i \in I}$ of functions $\beta_i^m : T_i \to \{l\} \cup B_i^m$ is an *m-equilibrium* if and only if, for any bidder i and any $t_i \in T_i$,

$$\forall b_i^m \in B_i^m \cup \{l\} : U_i(\beta_i^m(t_i), t_i, \beta^m) \geq U_i(b_i^m, t_i, \beta^m). \tag{23}$$

If, in addition, β_i^m is weakly increasing for every i, then the m-equilibrium is said *monotone*. The next proposition follows from Kakutani's fixed point theorem applied to each m-approximation game based on the single-crossing property implied by Theorem 1. The proof is the same as Athey's [1, Theorem 1] and hence omitted.

Proposition 3 *For any $m = 1, 2, \ldots,$ there exists a monotone m-equilibrium.*

By revealed preference, at any m-equilibrium a bidder never bids more than his expected payoff as a winner if he stands a positive probability of winning:

Lemma 6 *For any $m = 1, 2, \ldots,$ if $(\beta_i^m)_{i \in I}$ is an m-equilibrium then for any $i \in I$ and any $t_i \in T_i$ such that $\Pr\{\beta_i^m(t_i) > \max_{k \neq i} \beta_k^m(\boldsymbol{t}_k)\} > 0$, we have $\overline{W}_i(\beta_i^m(t_i), t_i, \beta^m) - \beta_i^m(t_i) \geq 0$.*

Proof Applying Ineq. (23) to the case $b_i^m = l$ and using Eqs. (10) and (22), we have

$$\Pr\left\{\beta_i^m(t_i) > \max_{k \neq i} \beta_k^m(\boldsymbol{t}_k)\right\} \left(\overline{W}_i(\beta_i^m(t_i), t_i, \beta^m) - \beta_i^m(t_i) - \overline{L}_i(\beta_i^m(t_i), t_i, \beta^m)\right) \geq 0.$$

By the hypothesis $\Pr\{\beta_i^m(t_i) > \max_{k \neq i} \beta_k^m(\boldsymbol{t}_k)\} > 0$, the term in the bracket "$(\cdots)$" is nonnegative. Then the conclusion of the lemma follows from $\overline{L}_i(\beta_i^m(t_i), t_i, \beta^m) \geq 0$, which is true because i can choose not to participate in the resale mechanism. ■

[15] Our approximation game is similar to that of Reny and Zamir [16]; they required the nonoverlapping condition (19) because their single crossing condition is not guaranteed at atomic bids. While single crossing condition holds even for atoms here by Theorem 1, we require (19) for the resale continuation game to be well-behaved, which would be a nonissue in the no-resale model of Reny and Zamir.

Not needed here is the other perturbation devised by Athey [1] and adopted by Reny and Zamir, that a bidder has to submit the losing bid l when his type belongs to $[0, 1/m)$. They need the perturbation to ensure a revealed-preference result. It would be redundant in this paper because our revealed-preference result is ensured by an upcoming notion of consequentiality, which is needed anyway for our no-tie argument.

5.2 Impossibility of Ties at the Limit

Given a monotone profile β of bid functions, call a serious bid b_* *consequential* if $\Pr\{\beta_k(\boldsymbol{t}_k) \leq b_*\} > 0$ for every bidder $k \in I$, and *inconsequential* if otherwise. A *tie* of β means a serious bid that is an atom for at least two distinct bidders according to their bid functions in β.

Theorem 2 (no tie) *If a sequence $(\beta^m)_{m=1}^\infty$ of monotone m-equilibria converges pointwise almost everywhere to a monotone profile β^*, then β^* admits no consequential tie.*

To prove Theorem 2, suppose to the contrary that β^* admits a consequential tie b_*. We shall derive a contradiction to the equilibrium property of the sequence $(\beta^m)_{m=1}^\infty$. As a preliminary, the next lemma provides a minute picture of the cluster of rivaling bids collapsing to the atom b_* as $m \to \infty$.

Lemma 7 *If a sequence $(\beta^m)_{m=1}^\infty$ of monotone profiles converges pointwise a.e. to a monotone profile β^* and if J is the set of bidders such that a serious bid b_* is an atom of β_j^* for all $j \in J$, then there exist subsequence $(\beta^{m_n})_{n=1}^\infty$ and sequence $(\delta_n)_{n=1}^\infty \to 0$ such that, with*

$$a_i := \sup\{t_i \in T_i : \beta_i^*(t_i) < b_*\}, \tag{24}$$

$$z_i := \sup\{t_i \in T_i : \beta_i^*(t_i) \leq b_*\}, \tag{25}$$

$$a_i^n := \inf\{t_i \in T_i : \beta_i^{m_n}(t_i) > b_* - \delta_n\}, \tag{26}$$

$$z_i^n := \sup\{t_i \in T_i : \beta_i^{m_n}(t_i) < b_* + \delta_n\} \tag{27}$$

for each i, we have:

$$\forall i \in J: \quad \forall t_i \in (a_i^n, z_i^n) : b_* - \delta_n < \beta_i^{m_n}(t_i) < b_* + \delta_n, \tag{28}$$

$$\forall i \in J: \quad \lim_{n\to\infty} \Pr\{t_i \in T_i \setminus (a_i^n, z_i^n) : b_* + \delta_n \leq \beta_i^{m_n}(t_i) \leq b_* + \delta_n + 2^{-m_n}\} = 0, \tag{29}$$

$$\forall i \in I: \quad a_i = \lim_{n\to\infty} a_i^n, \quad z_i = \lim_{n\to\infty} z_i^n, \tag{30}$$

$$\forall k \notin J: \quad \lim_{n\to\infty} \Pr\{t_k \in T_k : b_* - \delta_n \leq \beta_k^{m_n}(t_k) \leq b_* + \delta_n + 2^{-m_n}\} = 0. \tag{31}$$

Proof Appendix E.1. ■

With the $(\delta_n)_{n=1}^\infty$ in Lemma 7, the collapsing interval $(b_* - \delta_n, b_* + \delta_n)$ is the range of the β^{m_n}-bids for those types of bidder i in (a_i^n, z_i^n), says Ineq. (28). Along the subsequence $(\beta^{m_n})_{n=1}^\infty$, Eq. (29) says that the probability with which the types outside (a_i^n, z_i^n) would bid within $(b_* - \delta_n, b_* + \delta_n)$ vanishes, Eq. (30) says that (a_i^n, z_i^n) converges to (a_i, z_i), and Eq. (31) says that if β_k^* has no atom at b_* then the probability with which player k bids in $(b_* - \delta_n, b_i^n)$, with b_i^n being any bidder i's lowest grid point above $b_* + \delta_n$, goes to zero. Given the subsequence $(\beta^{m_n})_{n=1}^{\infty}$ identified in Lemma 7, for each n denote

$$\bar{\beta}^n := \beta^{m_n}.$$

By Eq. (31) and the consequentiality of b_*, we have

$$\forall k \notin J : \lim_{n\to\infty} \Pr\left\{\bar{\beta}_k^n(\boldsymbol{t}_k) < b_* - \delta_n\right\} > 0. \tag{32}$$

For any $n \in \{1, 2, \ldots\}$, any i, any $t_i^n \in T_i$ and any bids b_i^n and c_i^n in $B_i^{m_n}$ with $b_i^n > c_i^n$, the expected-payoff difference for a type-t_i^n bidder i caused by his bid increase from c_i^n to b_i^n in the m_n-equilibrium β^{m_n} is

$$\Delta U_i^n(t_i^n) := U_i(b_i^n, t_i^n, \bar{\beta}^n) - U_i(c_i^n, t_i^n, \bar{\beta}^n). \tag{33}$$

To prove Theorem 2 by contradiction, it suffices to find a bidder i and a sequence $(t_i^n, c_i^n, b_i^n)_{n=1}^{\infty}$ such that $\limsup_n \Delta U_i^n(t_i^n) > 0$ and, for any sufficiently large n, the $\bar{\beta}_i^n$-inverse-image of c_i^n is nondegenerate and contains t_i^n. Then for all sufficiently large n, $\Delta U_i^n(t_i^n) > 0$ and, with $\Delta U_i^n(\cdot)$ continuous (Lemma 5), the strict inequality extends to a neighborhood of t_i^n, which contradicts the fact that $\bar{\beta}^n$ constitutes an m_n-equilibrium.

To this end, decompose $\Delta U_i^n(t_i^n)$ into three parts (proved in Appendix E.2):

$$\Delta U_i^n(t_i^n) = \Delta W_i^n(t_i^n) - \Delta b^n + \Delta \Pi_i^n(t_i^n), \tag{34}$$

where

$$\begin{aligned}
\Delta W_i^n(t_i^n) &:= \Pr\left\{b_i^n > \max_{k\neq i} \bar{\beta}_k^n(\boldsymbol{t}_k)\right\}\left(\overline{W}_i(b_i^n, t_i^n, \bar{\beta}^n) - \overline{W}_i(c_i^n, t_i^n, \bar{\beta}^n)\right), && (35)\\
\Delta b^n &:= (b_i^n - c_i^n)\Pr\left\{b_i^n > \max_{k\neq i} \bar{\beta}_k^n(\boldsymbol{t}_k)\right\},\\
\Delta \Pi_i^n(t_i^n) &:= \Pr\left\{b_i^n > \max_{k\neq i} \bar{\beta}_k^n(\boldsymbol{t}_k) > c_i^n\right\}\left(\overline{W}_i(c_i^n, t_i^n, \bar{\beta}^n) - c_i^n - \overline{L}_i^n(t_i^n)\right), && (36)\\
\overline{L}_i^n(t_i^n) &:= \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid t_i^n, \bar{\beta}^n)\,\middle|\, b_i^n > \max_{k\neq i} \bar{\beta}_k^n(\boldsymbol{t}_k) > c_i^n\right].
\end{aligned}$$

Eq. (34) says that $\Delta U_i^n(t_i^n)$ consists of the *revenue effect* $\Delta W_i^n(t_i^n)$, *payment effect* Δb^n, and *pivotal effect* $\Delta \Pi_i^n(t_i^n)$, which includes $\overline{L}_i^n(t_i^n)$, the *winner's curse* in our context.

5.2.1 Step 1: Locating a Deviant Bidder

Recall the set J of tying rivals specified in Lemma 7. Pick an element $j \in J$ such that

$$\forall k \in J : a_j \leq a_k. \tag{37}$$

With $B_j^{m_n}$ discrete, there exists

$$c_j^n := \min \left\{ \bar{\beta}_j^n(t_j) : t_j \in \left(a_j^n, z_j^n\right) \right\}. \tag{38}$$

Lemma 8 $\lim_{n\to\infty} \Pr \left\{ c_j^n < \max_{k \in J \setminus \{j\}} \bar{\beta}_k^n(\boldsymbol{t}_k) \leq b_* + \delta_n \right\} = 0.$

Lemma 8 is proved in Appendix E.3. It can be understood from the viewpoint of those types of bidder j nearby a_j. If the lemma were not true, there would be a mass of rivaling bids within $(b_* - \delta_n, b_* + \delta_n)$ that outbid such types of bidder j, and the mass would not vanish along the sequence of the approximation equilibria. On one hand, with valuation nearly equal to a_j and with Ineq. (37), such types of bidder j would have almost zero gain from buying the good from these rival-types at resale, i.e., the winner's curse for such types of bidder j to jump over these rival-types is negligible. On the other hand, if such a low-value bidder j outbids these rival-types, he would profit from reselling to them, again due to Ineq. (37); with the mass of these rival-types nonvanishing, this expected profit is bounded away from zero. Both sides considered, bidder j with types nearby a_j would deviate to a bid slightly above $(b_* - \delta_n, b_* + \delta_n)$ if Lemma 8 does not hold.

For any $n = 1, 2, \ldots$ and any $i \in J \setminus \{j\}$, with c_j^n defined in Eq. (38), let

$$c_i^n := \max \left\{ \bar{\beta}_i^n(t_i) : t_i \in \left[0, \left(\bar{\beta}^n\right)_{i,\inf}^{-1} \left(c_j^n\right)\right) \right\}. \tag{39}$$

For any sufficiently large n, $\left[0, \left(\bar{\beta}^n\right)_{i,\inf}^{-1} \left(c_j^n\right)\right) \neq \varnothing$ due to Lemma 8 and the hypothesis that b_* is consequential; with $B_i^{m_n}$ discrete, c_i^n exists.

Since $J \setminus \{j\}$ is finite, there exists $i \in J \setminus \{j\}$ with $c_i^{n_\gamma} = \max_{k \in J \setminus \{j\}} c_k^{n_\gamma}$ for all γ in an infinite subsequence $(n_\gamma)_{\gamma=1}^\infty$. For this i, $\lim_{\gamma\to\infty} \Pr \left\{ c_i^{n_\gamma} < \max_{k \in J \setminus \{j,i\}} \bar{\beta}_k^{n_\gamma}(\boldsymbol{t}_k) < c_j^{n_\gamma} \right\} = 0$. Combining this with Lemma 8 and Eq. (39) and relabeling subsequence $(n_\gamma)_{\gamma=1}^\infty$, we have

$$\lim_{n\to\infty} \Pr \left\{ c_i^n < \max_{k \in J \setminus \{j\}} \bar{\beta}_k^n(\boldsymbol{t}_k) < b_* + \delta_n \right\} = 0. \tag{40}$$

Thus, as $n \to \infty$, the m_n-equilibrium bids from all players other than bidder j vanish from $(c_i^n, b_* + \delta_n)$. By $c_i^n < c_j^n$, the interval $(c_i^n, b_* + \delta_n)$ is almost exclusively occupied by the bids from bidder j with types in (a_j^n, z_j^n), which converges to the nondegenerate (a_j, z_j) since b_* is an atom of β_j^*. This coupled with Eq. (32) (consequentiality of b_*) implies

$$\lim_{n\to\infty} \Pr\left\{c_i^n < \max_{k\in I\setminus\{i\}} \bar{\beta}_k^n(\boldsymbol{t}_k) < b_* + \delta_n\right\} > 0. \tag{41}$$

By construction, $c_i^n < c_j^n < b_* + \delta_n$; by Eq. (40), the mass of i's bids in $(c_i^n, b_* + \delta_n)$ vanishes while, with $i \in J$, a nonvanishing mass of i's bids remains in $(b_* - \delta_n, b_* + \delta_n)$. Thus, for all large n, $c_i^n > b_* - \delta_n$ and hence

$$b_* - \delta_n < c_i^n < c_j^n < b_* + \delta_n. \tag{42}$$

By Eq. (39), the $\bar{\beta}_i^n$-inverse-image of c_i^n is nondegenerate. To complete the proof by contradiction, it suffices to prove existence of a sequence $(t_i^n)_{n=1}^\infty$ such that each t_i^n belongs to this inverse image and $\limsup_n \Delta U_i^n(t_i^n) > 0$, with $\Delta U_i^n(t_i^n)$ the expected-payoff difference rendered by the deviation from c_i^n to

$$b_i^n := \min\left\{b_i \in B_i^{m_n} : b_i \geq b_* + \delta_n\right\}. \tag{43}$$

To this end, we calculate the three components of $\Delta U_i^n(t_i^n)$ according to Eq. (34). Among them, the payment effect Δb^n is $O(\delta_n)$ (hence $O(1/n)$ by Lemma 7) because of Ineq. (42) and $b_i^n - c_i^n \leq 2^{m_n} + b_* + \delta_n - c_i^n$, which follows directly from Eq. (43). Thus, we need only to calculate the revenue effect $\Delta W_i^n(t_i^n)$ and pivotal effect $\Delta \Pi_i^n(t_i^n)$.

5.2.2 Step 2: The Revenue Effect of the Deviation

By a revealed-preference argument, one can prove $\Delta W_i^n \geq 0$ (Proposition 4, Appendix A.1.2). The next lemma asserts further that the revenue effect is bounded away from zero if bidder i has potential gain of trade with his rivals when he wins with the higher bid.

Lemma 9 *If $t_i^n \to_n t_i$ such that $0 < t_i < \max_{k\neq i} z_k$, then $\limsup_{n\to\infty} \Delta W_i^n(t_i^n) > 0$.*

Proof Appendix E.5. ■

By Eq. (30), z_k is the limit of the supremum z_k^n of bidder k's types that bid below b_i^n in the m_n-equilibrium. Hence the condition "$t_i^n \to_n t_i$ such that $t_i < \max_{k\neq i} z_k$" implies that, for all approximation equilibria sufficiently far along the sequence, bidder i can profit from reselling the good to his rivals if he wins with the bid b_i^n. By Eq. (41), the mass of rival-types surpassed by the bid increase does not vanish along the sequence. Hence the bid increase brings about a nonvainishing increase of resale probability and expected revenue.

5.2.3 Step 3: Pivotal Effect Case One: Bypassing the Middleman

Two cases need to be considered on the pivotal effect $\Delta\Pi_i^n(t_i^n)$. In the first case, bidder i's type is so high that, in the event of tying at b_* and he loses to bidder j, he buys the good nearly for sure from bidder j. Essentially a middleman, bidder j charges this type of i a price markup in addition to the period-one price. In making the bid increase thereby surpassing j, bidder i avoids paying the price markup, which constitutes the pivotal effect in this case.

More precisely, for any $k \in I$ and any $x \in T_k$, define

$$\mathcal{V}_{k,x}(t_k) := \begin{cases} t_k - (F_k(x) - F_k(t_k))/f_k(t_k) & \text{if } 0 \leq t_k \leq x \\ x & \text{if } t_k \geq x. \end{cases} \tag{44}$$

By Lemma 21 (Appendix E.4, due to Eq. (40)), when bidder j wins with a bid in the collapsing $(c_j^n, b_* + \delta_n)$, every losing bidder k's posterior virtual utility function converges to $\mathcal{V}_{k,z_k}$ as $n \to \infty$. Hence the precise meaning of our first case is that at the limit bidder i outranks everyone else in terms of $(\mathcal{V}_{k,z_k})_{k \neq j}$, as hypothesized in the next lemma.

Lemma 10 *If $t_i^n \to_n t_i$ such that $\mathcal{V}_{i,z_i}(t_i) \geq \max_{k \neq i} z_k$, then $\lim_{n\to\infty} \Delta\Pi_i^n(t_i^n) > 0$.*

Proof Appendix E.6. ■

Since the types of j that bid in $(c_j^n, b_* + \delta_n)$ would nearly for sure resell the good to bidder i when i's type happens to satisfy the hypothesis of the lemma, the expected payment extracted from such a high type of bidder i is larger than j's expected resale revenue by a nonvanishing margin, as i could be of low types according to j's posterior belief (Lemma 17, Appendix A.1.1). With j's expected resale revenue never below his period-one winning bid (Lemma 6), this nonvanishing margin implies a nonvanishing markup between the current price of the good and the expected resale price that the reseller j would impose on the high-type bidder i. This markup constitutes the pivotal effect of the bid increase.

5.2.4 Step 4: Pivotal Effect Case Two: Becoming the Middleman

Here comes the other case, where bidder i's type is not high enough to nearly for sure buy the good at resale from bidder j. Different than the previous case, the current price of the good could be higher than the resale price that j would charge i at resale in the event that bidder i's deviation is pivotal: Even if the revenue extracted from i is less than what j pays at period one, j can still profit from the revenues extracted from the other potential buyers, who are pressured to bid high given i's presence at resale.[16] Then the deviant bidder i suffers a winner's curse in the magnitude of the period-one price minus the lower price at resale.

The solution is to turn the table: In the same way that j's loss from dealing with i is balanced by j's revenues extracted from other bidders, i's winner's curse can be balanced by the revenues from the same clientele if i becomes the reseller instead of j. Denote

$$\Omega_i^n := \left\{ t_{-i} \in T_{-i} : \max_{k \notin \{i,j\}} \bar{\beta}_k^n(t_k) < b_i^n; c_i^n < \bar{\beta}_j^n(t_j) < b_i^n \right\}, \tag{45}$$

$$\psi_i^n(t_i^n) := \overline{W}_i\left(c_i^n, t_i^n, \bar{\beta}^n\right) - c_i^n - \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid t_i^n, \bar{\beta}^n) \mid \Omega_i^n\right]. \tag{46}$$

Hence Ω_i^n is the pivotal event of i's bid increase, and $\psi_i^n(t_i^n)$ his expected payoff from winning minus his winning bid and minus his winner's curse.

Lemma 11 *If $\bar{\beta}_i^n(t_i^n) = c_i^n$ for each n and $(t_i^n)_{n=1}^\infty$ converges, then*

$$\lim_{n\to\infty} \psi_i^n(t_i^n) \geq \lim_{n\to\infty} \mathbb{E}\left[\mathbf{1}\left[\boldsymbol{t}_j < \mathcal{V}_{i,z_i^n}(t_i^n)\right] \left(W_j\left(t_i^n, \boldsymbol{t}_{-(i,j)} \mid \bar{\beta}_j^n(\boldsymbol{t}_j), \boldsymbol{t}_j, \bar{\beta}^n\right) - \bar{\beta}_j^n(\boldsymbol{t}_j)\right) \Big| \, \Omega_i^n\right]. \tag{47}$$

Proof Appendix E.7. ■

Lemma 12 *There exists a sequence $(t_i^n)_{n=1}^\infty$ such that $\bar{\beta}_i^n(t_i^n) = c_i^n$ for each n and*

$$\limsup_{n\to\infty} \mathbb{E}\left[\mathbf{1}\left[\boldsymbol{t}_j < \mathcal{V}_{i,z_i^n}(t_i^n)\right] \left(W_j\left(t_i^n, \boldsymbol{t}_{-(i,j)} \mid \bar{\beta}_j^n(\boldsymbol{t}_j), \boldsymbol{t}_j, \bar{\beta}^n\right) - \bar{\beta}_j^n(\boldsymbol{t}_j)\right) \Big| \, \Omega_i^n\right] \geq 0. \tag{48}$$

Proof Appendix E.8. ■

To explain the two lemmas, let us temporarily pretend that, when j's bid is clustered around the tie, bidder i somehow knows exactly what j's bid b_j is equal to. Consider a resale mechanism M^n that i could offer if he wins at period one: First, i announces his own type t_i^n and then asks bidder j whether t_i^n is above the reserve price that j would have

[16] For example, suppose that in the continuation game where bidder j is the reseller, $t_j = 2$, $\boldsymbol{t}_i$ is uniformly distributed on $[0, 4]$, and $\boldsymbol{t}_k$ uniformly distributed on $[0, 10]$. In j's optimal resale mechanism, the maximum of bidder i's expected payment (when $t_i = 4$) is equal to

$$\frac{6}{10} \times 3 + \int_6^7 (t_k - 3) dt_k/10 = 2.15,$$

while the reseller j's expected payoff equals

$$\frac{3}{4} \times \frac{6}{10} \times 2 + \frac{1}{4} \times \frac{6}{10} \times 3 + \frac{3}{4} \times \frac{4}{10} \times 6 + \int_3^4 \int_{t_i+3}^{10} (t_i + 3) \frac{dt_k}{10} \frac{dt_i}{4} + \int_6^7 \int_{t_k-3}^{4} (t_k - 3) \frac{dt_i}{4} \frac{dt_k}{10} \approx 3.76.$$

Thus, at period one, it is possible for bidder j to submit a bid strictly between 2.15 and 3.76.

offered i had j been the winner, which implies that j would always resell the good had j won. If bidder j says No, then i offers resale to all bidders via the Myerson auction that i should have chosen on the $\bar{\beta}^n$-equilibrium path. If bidder j answers Yes, by contrast, i offers resale to all but bidder j via j's resale mechanism, where i's own announced type, together with the losing bidders', are discounted to their virtual utilities. Here i can replicate j's resale mechanism because of our temporary assumption that i knows j's bid b_j. Reseller i's uncertainty about j's type makes no difference, because i excludes j in this case.

While the mechanism M^n is suboptimal to i, it generates enough expected revenue to cover the winner's curse and winning bid. To see why, note that the winner's curse is null if bidder j, presumed honest, answers No to i's question. In that case, bidder i's net gain is just his expected revenue as a reseller minus his winning bid. Since his resale mechanism in that case coincides with the Myerson auction that he should have chosen on path, the expected revenue it generates is the same as his on-path expected revenue, which can cover the winning bid by a revealed-preference argument (Lemma 6).

Thus, consider the case where bidder j answers Yes to i's question. In that case, i's resale mechanism M^n either keeps the good to i himself or resells the good to some bidder k other than j. Similarly, had bidder i lost to j at period one then j would resell the good to either bidder i or some other losing bidder k but would never keep the good to j herself. The events for these final outcomes are identical between M^n and j's resale mechanism, since the two mechanisms coincide when j honestly answers Yes. Let us calculate i's gain and loss from outbidding j in these two events:

final owner	gain	foregone trade with j	current price	net gain
i	i's use value t_i^n	$t_i^n - p_{ij}^n$	c_i^n	$p_{ij}^n - c_i^n$
k	p_{ki}^n	0	c_i^n	$p_{ki}^n - c_i^n$

Here p_{ij}^n denotes the resale price that i would need to pay j had j won, and p_{ki}^n the resale price at which k buys from i in M^n. Since M^n replicates j's resale mechanism, $p_{ki}^n = p_{kj}^n$. Thus, whether the final owner is i himself or some $k \notin \{i, j\}$, i's net gain from outbidding j is nearly the same as j's profit had j won (with $c_i^n \approx b_* \approx b_j$), which is nonnegative by a revealed-preference argument for bidder j.

In sum, whether j answers Yes or No to i's question, i's payoff from outbidding j can nearly offset the winner's curse (foregone gain of buying from j) and the current price. This is the combined implication of Ineqs. (47) and (48), where the indicator function

$\mathbf{1}\left[\boldsymbol{t}_j < \mathcal{V}_{i,z_i^n}(t_i^n)\right]$ corresponds to j's affirmative answer.

Two problems in the above heuristic argument need to be repaired. First, the argument was based on a false assumption that i somehow knows j's bid $\bar{\beta}_j^n(\boldsymbol{t}_j)$. However, removing this assumption does not upset our conclusion. By Eq. (40), when i's bid increase is pivotal, bidder j's bid $\bar{\beta}_j^n(\boldsymbol{t}_j)$ ranges in the interval $(c_i^n, b_* + \delta_n)$ where the others rarely bid, hence the resale mechanism selected by j as a reseller stays mostly constant. Therefore, i can nearly replicate j's mechanism with the pretended winning bid b_j being any element in this interval.

The second problem is that each player accounts its own profits and loss based on its private information, hence nonnegative expected profit from j's viewpoint need not imply nonnegative expected profit conditional on the realized type of bidder i. Lemma 12 solves this problem by observing that there exist types t_i^n of bidder i conditional on which j's expected profit is nonnegative. In order for such t_i^n to be those whose $\bar{\beta}_i^n$-bids equal c_i^n, essentially the highest among i's bids that belong to the tying cluster, we need such t_i^n to exist at the high end of bidder i's posterior support. That is ensured by comparative statics of the Myerson auction (Lemma 19, Appendix A.2.2).

5.2.5 Step 5: Completing the Proof of Theorem 2

There are only two possible cases: either (i) $z_i < \max_{k \neq i} z_k$ or (ii) $z_i \geq \max_{k \neq i} z_k$.

In Case (i), by Lemma 12, there exists a sequence $(t_i^n)_{n=1}^{\infty}$ such that $\bar{\beta}_i^n(t_i^n) = c_i^n$ for each n and Ineq. (48) holds. Extracting a converging subsequence if necessary, we may assume without loss of generality that $t_i^n \to_n t_i$ for some t_i. Then Lemma 11 says that Ineq. (47) holds. Combining both inequalities we have

$$\limsup_{n\to\infty} \Delta\Pi_i^n(t_i^n) = \limsup_{n\to\infty} \Pr(\Omega_i^n) \limsup_{n\to\infty} \psi_i^n(t_i^n) \geq 0. \tag{49}$$

For any n, with $c_i^n < b_* + \delta_n$ by Ineq. (42), $t_i^n \leq z_i^n$. Hence $t_i \leq z_i < \max_{k\neq i} z_k$. Thus, Lemma 9 implies $\limsup_{n\to\infty} \Delta W_i^n(t_i^n) > 0$. Plugging into Eq. (34) this strict inequality, as well as Ineq. (42) and Eq. (49), we have $\limsup_{n\to\infty} \Delta U_i^n(t_i^n) > 0$.

In Case (ii), where $z_i \geq \max_{k\neq i} z_k$, Lemma 10 implies that $\lim_{n\to\infty} \Delta\Pi_i^n(z_i) > 0$. Plugging this into Eq. (34) and noting $\Delta W_i^n(z_i) \geq 0$ (Proposition 4, Appendix A.1.2) and Eq. (42), we obtain $\lim_{n\to\infty} \Delta U_i^n(z_i) > 0$. With $\Delta U_i^n(t_i)$ continuous in t_i (Lemma 5), there exists $\alpha < z_i$ such that $\lim_{n\to\infty} \Delta U_i^n(t_i') > 0$ for all $t_i' \in (\alpha, z_i]$. By Eq. (40), the distance between z_i^n and the supremum of the inverse image $\left(\bar{\beta}^n\right)_i^{-1}(c_i^n)$ converges to zero;

thus, with $z_i^n \to_n z_i$ by Eq. (30), this supremum converges to z_i. Hence we can pick a sequence $(t_i^n)_{n=1}^\infty$ such that $t_i^n \in \left(\bar{\beta}^n\right)_i^{-1}(c_i^n)$ for each n and $t_i^n \to_n z_i$. Then for all sufficiently large n, $t_i^n \in (\alpha, z_i]$ and hence $\limsup_{n\to\infty} \Delta U_i^n(t_i^n) > 0$. Therefore, the desired contradiction $\limsup_{n\to\infty} \Delta U_i^n(t_i^n) > 0$ is obtained, which completes the proof of Theorem 2. ■

Slightly modifying the construction of c_i^n, we can extend the above proof to obtain—

Corollary 1 *If $b_* > r$ then b_* is not a consequential atom of β^*.*

Proof Appendix E.9. ■

6 Payoff Security at the Limit

Recall our objective of showing that Ineq. (23) converges to the equilibrium condition for the limit profile β^*, i.e., $U_i(\beta_i^*(t_i), t_i, \beta^*) \geq U_i(b_i, t_i, \beta^*)$ for all possible bids b_i. While Theorem 2 has ensured that the left-hand side of (23) does converge to $U_i(\beta_i^*(t_i), t_i, \beta^*)$ for almost every t_i (since the probability which which $\beta_i^{m_n}(t_i)$ converges to an atom of β_{-i}^* is zero), one would still wish to show that for any bid b_i, $U_i(b_i, t_i, \beta^*)$ is the limit of a sequence of $(U_i(b_i^m, t_i, \beta^*))_{m=1}^\infty$ such that each b_i^m is an unchosen option for i in the approximation equilibrium β^m. But such convergence is impeded by a case where b_i is an atom of β^*. In that case, even if there is a sequence $(b_i^m)_{m=1}^\infty \to b_i$ each of which is an admissible bid in the corresponding m-approximation game, the right-hand side of (23) still does not converge to the discontinuous point $U_i(b_i, t_i, \beta^*)$. In the fixed-point literature, this problem is solved by a payoff-security condition saying that bidder i can do nearly as well by bidding slightly above the atom b_i. This condition, however, relies on a monotonicity assumption that may fail with resale, as explained in the Introduction.

Our solution is a payoff-security theorem saying that a bidder, instead of bidding at an atom, can do nearly as well by either abstention or bidding slightly above the atom. This is nontrivial because being hurt from bidding above an atom does not automatically imply that bidding below it does not hurt. For instance, say the reserve price r is an atom of a bidder $j \neq i$ and bidder i's type is way above r so that i gets a big positive payoff if i wins by bidding the atom r. Suppose that there is a large probability with which all bidders other than i submit the losing bid l, so that if i bids below r (by submitting l) then he gets zero payoff with a large probability, rendering him worse-off than bidding at the atom r. On the other hand, bidding above r may also be worse-off when i expects a large gain from buying the good at resale from the r-bidding types of j.

Theorem 3 (payoff security) *If a sequence $(\beta^m)_{m=1}^{\infty}$ of monotone m-equilibria converges pointwise a.e. to a monotone profile β^*, then for any bidder i, any $t_i \in T_i$ and any atom b_* of β^*_{-i}, either (i) $U_i(l, t_i, \beta^*) \geq U_i(b_*, t_i, \beta^*)$ or (ii) $\lim_{n\to\infty} U_i(b^n, t_i, \beta^*) \geq U_i(b_*, t_i, \beta^*)$ for any sequence $(b^n)_{n=1}^{\infty}$ such that $b^n \downarrow b_*$ and none in the sequence is an atom of β^*.*

The proof of the theorem is facilitated by some comparative static properties of the Myerson resale auction and a general feature of Theorem 1, which asserts an increase difference result that allows for atoms. A main part of the proof is illustrated by Figure 2,

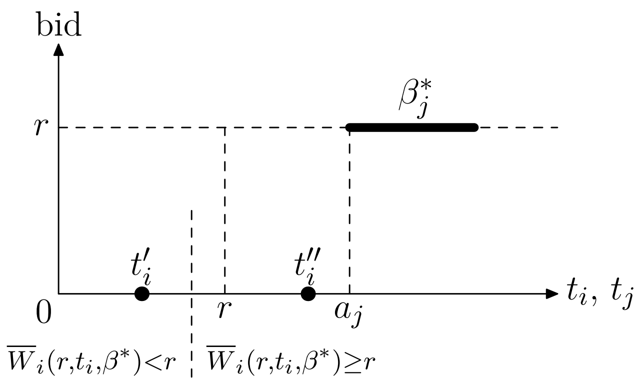

Figure 2: Proving the payoff-security theorem

where the reserve price r is a consequential atom for bidder j and j only. This atom cannot be ruled out in the manner of the no-tie theorem, because anyone else who bids below r submits l, bounded away from r, would suffer a significant increase of payments in jumping up to r or above. To prove payoff security in this case, we first observe that a_j, the infimum of bidder j's r-bidding types, cannot fall below the reserve price:

Lemma 13 *If r is a consequential atom of β^*_j then* $\inf\{t_j : \beta^*_j(t_j) = r\} \geq r$.

Proved in Appendix F.1, Lemma 13 can be understood from the viewpoint of the type a_j of bidder j. If, contrary to the lemma, $a_j < r$, then a_j pays a price r above his use value and hence needs to recover the loss from resale revenues. For that to happen, he needs some other bidders to abstain from the initial auction up to a type above r in the m-equilibria with large m. Then the supremum z_i among such abstaining types would deviate to a bid slightly above r: While the deviation increases his payment from zero to r, the cost is more than outweighed by the saved resale price charged by j.

By Lemma 13, one can show that there exist types of bidder i, such as the t_i'' in Figure 2, for which $t_i'' \le a_j$ and the expected payoff $\overline{W}_i(r, t_i'', \beta^*)$ from winning at price r does not fall below r. Payoff security for such t_i'' is at hand: Instead of bidding at the atom r, t_i'' can bid slightly above r. With $t_i'' \le a_j$, the winner's curse rendered by this bid increase is negligible, as in Lemma 8. Coupled with the fact $\overline{W}_i(r, t_i'', \beta^*) \ge r$, this implies nearly nonnegative pivotal effect. Thus, the expected-payoff difference caused by this slight increase of bids is nearly nonnegative provided that the revenue effect is also nearly nonnegative. Then our increasing difference theorem implies that the expected-payoff difference cannot become more negative when t_i'' becomes higher, even if high enough to cause a large winner's curse. The other case, where i's type falls down to say t_i' in Figure 2, is easy. There, $\overline{W}_i(r, t_i', \beta^*) < r$ and one readily sees that the type t_i' would rather submit the losing bid l than r. Payoff security with respect to r is hence established for all types of i.

In the above reasoning, the only gap to fill is the claim that the revenue effect of the bid increase is nearly nonnegative, as asserted by the next lemma. It is a nontrivial fact because the lower bid r here is an atom, rendering the resale environment partially irregular for a reseller who won after bidding r, which Appendix A.2 handles in detail.

Lemma 14 *For any monotone profile β, if b is an atom of β_j and β_j only and if $(b^n)_{n=1}^{\infty}$ converges to b from above and none in the sequence is an atoms of β, then for any bidder $i \neq j$ and any $t_i \in T_i$, $\lim_{n\to\infty} \overline{W}_i(b^n, t_i, \beta) \ge \overline{W}_i(b, t_i, \beta)$.*

Proof By definition, $\overline{W}_i(b, t_i, \beta)$ is i's optimal expected payoff as a winner based on the posteriors $(F_k(\cdot \mid i, b, \beta))_{k\neq i}$ defined by Eq. (68), which is a semi-regular resale environment defined in §A.2 due to b being an atom of β_j. Thus, $\overline{W}_i(b, t_i, \beta) = \Phi(0, t_{i_*})$, where Φ is defined in Eq. (62), with $\left(i, j, \beta^{-1}_{j,\inf}(b), \beta^{-1}_{j,\sup}(b)\right)$ playing the role $(i_*, i, \alpha_i, \zeta_i)$ there. By contrast, $\overline{W}_i(b^n, t_i, \beta)$ is i's expected payoff as a winner when his winning bid b^n avoids the atoms of β_{-i}, which constitutes a regular resale environment defined in §A.1. For each potential buyer $k \neq i$, when $b^n \downarrow b$, the supremum of the support of $\boldsymbol{t}_k$ converges to $\beta^{-1}_{k,\sup}(b)$ and the posterior distribution converges to the regular $F_{k,\beta^{-1}_{k,\sup}(b)}$ defined in Eq. (50). Thus,

$$\lim_{n\to\infty} \overline{W}_i(b^n, t_i, \beta) = \int_0^{\beta^{-1}_{j,\sup}(b)} \varphi_j(t_j, t_i) F_{j,\beta^{-1}_{j,\sup}(b)}(dt_j).$$

Then the lemma follows from Corollary 4. ■

The other case that needs handling in the proof of the theorem is that the atom is inconsequential. In this case, if our bidder i cannot unilaterally alter the inconsequentiality of the atom, then the atom is effectively the same as the losing bid l and payoff security for i follows. By contrast, if an atom b_* is inconsequential only because i's β_i^*-bid is above b_* almost surely, then when i bids b_* instead of abiding by β_i^*, the bid b_* would have positive winning probability and cause possible discontinuity in i's expected payoff. This possibility is eliminated by the next lemma.

Lemma 15 *If a serious bid b_* is an inconsequential atom of β^* then there are at least two bidders such that, for each k of the two, b_* is not an atom of β_k^* and $\Pr\{\beta_k^*(\boldsymbol{t}_k) > b_*\} = 1$.*

The proof, in Appendix F.2, is similar in spirit to that of Lemma 10: If bidder i is the only one who bids above the atom almost surely, then in some m-equilibria with sufficiently large m, there would be sufficiently high types of a bidder j who would almost always outrank his rivals in the resale mechanism offered by those types of bidder i who would have won without j's deviation. These types of i, if undefeated, act merely as middlemen for j and impose on j a price markup (Lemma 17, Appendix A.1.1), which bidder j could have bypassed with a higher bid. Different from Lemma 10, however, these types of bidder i do not constitute a nonvanishing mass. Therefore, much of the proof of the lemma is to fine-tune the magnitude of j's deviation so that his expected net gain is strictly positive.

Proof of Theorem 3 The atom b_* is either consequential or not. First, suppose that it is consequential. Then, by Corollary 1, $b_* = r$ and it is the atom of only one bidder, say j. Furthermore, by Lemma 13, $a_j := \inf\left\{t_j : \beta_j^*(t_j) = r\right\} \geq r$. Pick any $i \neq j$ and let

$$A_i := \left\{t_i \in T_i : \overline{W}_i(r, t_i, \beta^*) - r < 0\right\}.$$

By monotonicity and continuity of $\overline{W}_i(r, \cdot, \beta^*)$ (the envelope equation in Lemma 5), A_i is an interval starting from zero and ends at $\sup A_i$, and $\overline{W}_i(r, t_i, \beta^*) - r \geq 0$ for all $t_i \geq \sup A$. Furthermore, with the facts $\overline{W}_i(r, t_i, \beta^*) \geq t_i$ and $a_j \geq r$, we have $\sup A_i \leq r \leq a_j$.

For any $t_i < \sup A_i$, $\overline{W}_i(r, t_i, \beta^*) - r < 0$ impies, by $\overline{L}_i \geq 0$ (individual rationality at resale) and Eq. (10), that $U_i(l, t_i, \beta^*) > U_i(r, t_i, \beta^*)$, which is alternative (i) claimed by the theorem. For any $t_i \geq \sup A_i$, we prove that alternative (ii) is true. Thus, pick any sequence

$(\epsilon_n)_{n=1}^\infty \downarrow 0$ such that for none of n is $r+\epsilon_n$ an atom of β^*. For each n consider a bid increase from r to $r+\epsilon_n$. It suffices to prove that the expected-payoff difference

$$\Delta U_i(t_i) := U_i(r+\epsilon_n, t_i, \beta^*) - U_i(r, t_i, \beta^*)$$

rendered by this bid increase, with others abiding by β^*, is no less than $-O(\epsilon_n)$. We need only to prove the claim for the case $t_i = \sup A_i$, then Theorem 1 extends the inequality to all higher types. To prove $\Delta U_i(\sup A_i) \geq -O(\epsilon_n)$, recall the fact $\sup A_i \leq a_j$ proved above. Thus, in the event that the reseller is bidder j with types t_j such that $\beta_j^*(t_j) = r$, $t_j \geq a_j$. Hence the winner's curse $\tilde{L}_i(\sup A, r+\epsilon_n, r, \beta^*) = 0$. Coupled with the fact $\overline{W}_i(r, \sup A_i, \beta^*) - r \geq 0$, this implies the pivotal effect

$$\Delta\Pi_i(\sup A_i) = \Pr\{\text{pivotal}\}\left(\overline{W}_i(r, \sup A_i, \beta^*) - r - \tilde{L}_i(\sup A, r+\epsilon_n, r, \beta^*)\right) \geq 0.$$

We also have $\Delta W_i(\sup A_i) \geq -O(\epsilon_n)$, by Lemma 14, and $\Delta b = O(\epsilon_n)$. Thus, $\Delta U_i(\sup A_i) \geq -O(\epsilon_n)$ via Eq. (34), as desired.

Second, consider the case where the atom b_* is inconsequential. By Lemma 15, for the bidder $i \neq j$ under our consideration, there exists a $k \neq i$ for whom $\beta_k^* > b_*$ almost surely. Thus, for bidder i, bidding b_* loses for sure, same as submitting the losing bid l. Hence $U_i(l, t_i, \beta^*) \geq U_i(b_*, t_i, \beta^*)$ for all t_i, which is alternative (i) claimed by the theorem. ■

7 Equilibrium of the Original Game

Since the no-tie and payoff-security theorems have essentially resolved the issue of atoms, to complete the existence proof we need only to demonstrate the convergence of a bidder's expected payoff at non-atom bids, asserted by the next lemma.

Lemma 16 *For any $i \in I$, any $t_i \in T_i$ and any sequence $(b_i^m)_{m=1}^\infty$ converging to b_i such that $b_i^m \in B_i^m$ for each m, if b_i is not an atom of β_{-i}^* then $\lim_{m\to\infty} U_i(b_i^m, t_i, \beta^m) = U_i(b_i, t_i, \beta^*)$.*

Proved in Appendix G, the lemma is mainly due to the convergence of a bidder's expected payoffs at resale, whose convergence is in turn due to the convergence of posterior virtual utilities when the winning bid is not an atom.

Theorem 4 (existence) *For any reserve price $r \geq 0$, the auction-resale game defined in §2.1 admits a monotone perfect Bayesian equilibrium; furthermore, at this equilibrium,* *any bidder's period-one bid that is above r and can win with strictly positive probability is a strictly increasing function of the bidder's use value.*

Proof By Proposition 3, there exists a sequence $(\beta^m)_{m=1}^{\infty}$ such that every β^m satisfies Ineq. (23). Taking a convergent subsequence and relabeling if necessary, we can assume without loss that $\beta^m \to \beta^*$ a.e., with β_i^* weakly increasing for each bidder i. Corollary 1 says that the consequential serious bids according to β^* are strictly increasing in the bidders' types except possibly the reserve price. Thus, by Lemma 4, it suffices to show that β^* constitutes a Nash equilibrium with respect to the interim expected payoff functions $(U_i(\cdot, \cdot, \beta^*))_{i \in I}$. By Theorem 2, the left-hand side of Ineq. (23) converges to $U_i(\beta_i^*(t_i), t_i, \beta^*)$ for almost every t_i. For any bid $b_i \in \{l\} \cup [r, \infty)$, we may assume without loss that b_i is not an atom of β_{-i}^*. That is because any atomic bid, by Theorem 3, can be replaced either by l or a sufficiently close bid b_i' that is not one of the at most countably many atoms of β_{-i}^* and, by Lemma 16 and the denseness of $\cup_m B_i^m$, b_i' in turn can be replaced by an element of B_i^m that, for sufficiently large m, is sufficiently close to the atomic bid. With the alternative bid b_i not an atom of β_{-i}^*, there exists a sequence $(b_i^m)_{m=1}^{\infty} \to_m b_i$ such that the right-hand side of Ineq. (23) converges to $U_i(b_i, t_i, \beta^*)$. Then the desired best-reply condition is obtained:

$$U_i(\beta_i^*(t_i), t_i, \beta^*) = \lim_{m \to \infty} U_i(\beta_i^m(t_i), t_i, \beta^m) \geq \lim_{m \to \infty} U_i(b_i^m, t_i, \beta^m) = U_i(b_i, t_i, \beta^*). \quad \blacksquare$$

8 Conclusion

Fixed-point approaches have been foundational to theoretical investigations of discontinuous games especially certain auction mechanisms. Incorporation of post-auction resale into such approaches is not only realistically relevant but also theoretically compelling, because, as noted in the literature, resources can be misallocated in certain asymmetric auctions, triggering the incentive for resale. The possibility of resale brings about new challenges to the fixed-point approaches. The value-correlation across bidders, previously assumed exogenous, becomes endogenously determined by resale, which is itself endogenous. The discontinuity problem of tying bids gets compounded to the discontinuity of post-auction beliefs and that of the payoffs at resale. Yet these challenges turn out to be surmountable, as demonstrated in this paper, which has extended the fixed-point approach beyond its previous confines of treating an auction as an isolated event.

To capture its endogenous nature, this paper models resale by assuming that the winning bidder gets to choose any resale mechanism, hence at equilibrium resale is offered through the Myerson auctions. The extent to which the properties of such resale mechanisms, instrumental to our existence proof, may be generalized to other resale settings is left for future investigations. Nevertheless, there is a merit, at least for the first endeavor, to endogenize resale mechanisms as in our model. It shows us the power of mechanism design, as a modeling technique, to pin down resale mechanisms among the myriad of secondary-market arrangements often hard to observe. Just as the rational choice axiom reduces individual behaviors to regularity, the endogenous treatment of resale mechanisms generates subtle comparative statics in equilibrium with forward-looking bidding behaviors.

A Comparative Statics of the Myerson Auction

The properties of Myerson's [13] optimal auction proved here are mainly about the monotonicity of the seller's expected payoff with respect to some aspects of the bidder-type distributions. However intuitive they might sound, these properties are not trivial to prove. Different from the case in the received literature, monotonicity does not follow from the affiliation inequalities in Milgrom and Weber [12], because the ex post revenue need not be nondecreasing in the bidders' realized types, as explained in the Introduction.

In our context, the Myeson auction corresponds to the equilibrium resale mechanism selected by a reseller, and the distributions the post-auction beliefs. Let $i_* \in I$ denote this (re)seller and $I \setminus \{i_*\}$. For any $i \in I \setminus \{i_*\}$, assume, on the support of the prior distribution F_i, that F_i has strictly positive density f_i and the prior virtual utility $t_i - (1 - F_i(t_i))/f_i(t_i)$ is strictly increasing in t_i.

A.1 Regular Resale Environments

A *regular* resale environment is characterized by a vector $\zeta := (\zeta_i)_{i \in I \setminus \{i_*\}}$ such that, for any $i \in I \setminus \{i_*\}$ the distribution of i's type $\boldsymbol{t}_i$ is an F_{i,ζ_i} derived from the prior F_i via

$$F_{i,\zeta_i}(t_i) := F_i(t_i)/F_i(\zeta_i) \tag{50}$$

for all t_i in the support $[0, \zeta_i]$, and likewise for the density f_{i,ζ_i}. Define the posterior virtual utility $\mathcal{V}_{i,\zeta_i}(t_i)$ by Eq. (44) if $t_i \le \zeta_i$ and by $\mathcal{V}_{i,\zeta_i}(t_i) := \zeta_i$ if $t_i \ge \zeta_i$. Then $\mathcal{V}_{i,\zeta_i}$ is strictly increasing and continuous on $[0, \zeta_i]$. For any $i \neq i_*$ and any $t_{-i} := (t_{i_*}, (t_k)_{k \notin \{i, i_*\}})$, denote

$$v_i(t_{-i}) := \max \left\{ t_{i_*}, \max_{k \notin \{i, i_*\}} \mathcal{V}_{k,\zeta_k}(t_k) \right\}. \tag{51}$$

Let $\mathbb{M}(\zeta, t_{i_*})$ denote the Myerson auction defined by $(\mathcal{V}_{k,\zeta_k})_{k \neq i_*}$, which for each realized type profile (t_{i_*}, t_{-i_*}) sells only to a bidder k for whom $\mathcal{V}_{k,\zeta_k}(t_k) \geq v_k(t_{-k})$ at price $\mathcal{V}^{-1}_{k,\zeta_k}(v_k(t_{-k}))$ and charges everyone else zero price. Denote $R(\zeta, t_{i_*}, t_{-i_*})$ for seller i_*'s ex post payoff ("revenue") generated by mechanism $\mathbb{M}(\zeta, t_{i_*})$ when t_{-i_*} is the realized type profile across $i \in I \setminus \{i_*\}$. Let

$$\overline{R}(\zeta, t_{i_*}) := \mathbb{E}\left[R(\zeta, t_{i_*}, \boldsymbol{t}_{-i_*}) \mid \boldsymbol{t}_{-i_*} \leqq \zeta_{-i_*}\right],$$

where, for any points x and y in the same euclidean space, $x \leqq y$ means $x_k \leq y_k$ for each coordinate k, and $x \nleqq y$ means "not $x \leqq y$". With every $\mathcal{V}_{k,\zeta_k}$ strictly increasing on $[0, \zeta_k]$, the seller's optimization problem belongs to the regular case of Myerson [13]. Thus,

$$\overline{R}(\zeta, t_{i_*}) = \mathbb{E}\left[\max \left\{ t_{i_*}, \max_{k \neq i_*} \mathcal{V}_{k,\zeta_k}(\boldsymbol{t}_k) \right\} \middle| \boldsymbol{t}_{-i_*} \leqq \zeta_{-i_*}\right]. \tag{52}$$

A.1.1 An Upper Bound of Expected Revenues

This upper bound is the expected payment made by the highest possible bidder-type.

Lemma 17 *If $i \in I \setminus \{i_*\}$ and $\zeta_i = \max_{k \neq i_*} \zeta_k > t_{i_*}$, then*

$$\overline{R}(\zeta, t_{i_*}) < \mathbb{E}\left[\mathcal{V}^{-1}_{i,\zeta_i}\left(\max \left\{ t_{i_*}, \max_{k \notin \{i, i_*\}} \mathcal{V}_{k,\zeta_k}(\boldsymbol{t}_k) \right\}\right) \middle| \boldsymbol{t}_{-(i,i_*)} \leqq \zeta_{-(i,i_*)}\right]. \tag{53}$$

Proof By Eqs. (51) and (52),

$$\begin{aligned}
\overline{R}(\zeta, t_{i_*}) &= \mathbb{E}\left[\max\left\{\mathcal{V}_{i,\zeta_i}(\boldsymbol{t}_i), v_i(t_{i_*}, \boldsymbol{t}_{-(i,i_*)})\right\} \middle| \boldsymbol{t}_{-i_*} \leqq \zeta_{-i_*}\right] \\
&= \underbrace{\mathbb{E}\left[v_i(t_{i_*}, \boldsymbol{t}_{-(i,i_*)}) \mathbf{1}\left[v_i(t_{i_*}, \boldsymbol{t}_{-(i,i_*)}) \geq \mathcal{V}_{i,\zeta_i}(\boldsymbol{t}_i)\right] \mid \boldsymbol{t}_{-i_*} \leqq \zeta_{-i_*}\right]}_{=:X} \\
&\quad + \underbrace{\mathbb{E}\left[\mathcal{V}_{i,\zeta_i}(\boldsymbol{t}_i) \mathbf{1}\left[v_i(t_{i_*}, \boldsymbol{t}_{-(i,i_*)}) < \mathcal{V}_{i,\zeta_i}(\boldsymbol{t}_i)\right] \mid \boldsymbol{t}_{-i_*} \leqq \zeta_{-i_*}\right]}_{=:Y}.
\end{aligned}$$

First, we calculate X. For any t_{-i_*} in the integration domain of X, the hypothesis $\zeta_i \geq \max\{t_{i_*}, \max_{k \notin \{i,i_*\}} \zeta_k\}$ implies $v_i(t_{i_*}, t_{-(i,i_*)}) \leq \zeta_i$ and hence $\mathcal{V}^{-1}_{i,\zeta_i}(v_i(t_{i_*}, t_{-(i,i_*)})) \geq v_i(t_{i_*}, t_{-(i,i_*)})$. Furthermore, there is a positive-measure subset of the integration domain in which this inequality is strict: When t_{-i_*} is nearly zero so that every bidder's virtual utility is negative, $v_i(t_{i_*}, t_{-(i,i_*)}) = t_{i_*}$ while $\mathcal{V}_{i,\zeta_i}^{-1}(v_i(t_{i_*}, t_{-(i,i_*)})) = \mathcal{V}_{i,\zeta_i}^{-1}(t_{i_*})$, strictly larger than t_{i_*} by Eq. (44). This subset is of positive measure in T_{-i_*} since the priors have no gap. Thus,

$$X < \mathbb{E}\left[\mathcal{V}_{i,\zeta_i}^{-1}(v_i(t_{i_*}, \boldsymbol{t}_{-(i,i_*)}))\mathbf{1}\left[v_i(t_{i_*}, \boldsymbol{t}_{-(i,i_*)}) \geq \mathcal{V}_{i,\zeta_i}(\boldsymbol{t}_i)\right] \mid \boldsymbol{t}_{-i_*} \leqq \zeta_{-i_*}\right]. \tag{54}$$

To calculate Y, denote $q_i(t_{-i_*}) := \mathbf{1}\left[v_i(t_{i_*}, t_{-(i,i_*)}) < \mathcal{V}_{i,\zeta_i}(t_i)\right]$, the probability with which bidder i buys the good at player i_*'s mechanism $\mathbb{M}(\zeta)$. Denote

$$\overline{q}_i(t_i) := \mathbb{E}\left[q_i(t_i, \boldsymbol{t}_{-(i,i_*)}) \mid \boldsymbol{t}_{-(i,i_*)} \leqq \zeta_{-(i,i_*)}\right].$$

By Eq. (44),

$$Y = \mathbb{E}\left[\left(\boldsymbol{t}_i - \frac{1 - F_{i,\zeta_i}(\boldsymbol{t}_i)}{f_{i,\zeta_i}(\boldsymbol{t}_i)}\right) q_i(\boldsymbol{t}_i, \boldsymbol{t}_{-(i,i_*)}) \middle| \boldsymbol{t}_{-i_*} \leqq \zeta_{-i_*}\right] = \int_0^{\zeta_i} \overline{q}_i(t_i)\left(t_i - \frac{1 - F_{i,\zeta_i}(t_i)}{f_{i,\zeta_i}(t_i)}\right) f_{i,\zeta_i}(t_i)dt_i.$$

Going through the integration-by-parts routine in reverse order, we have

$$\begin{aligned}
Y &= \int_0^{\zeta_i} t_i\overline{q}_i(t_i)f_{i,\zeta_i}(t_i)dt_i - \int_0^{\zeta_i}\int_{t_i}^{\zeta_i} \overline{q}_i(t_i)f_{i,\zeta_i}(t_i')dt_i'dt_i \\
&= \int_0^{\zeta_i} t_i\overline{q}_i(t_i)f_{i,\zeta_i}(t_i)dt_i - \int_0^{\zeta_i}\int_0^{t_i} \overline{q}_i(t_i)dt_i f_{i,\zeta_i}(t_i')dt_i' \\
&= \int_0^{\zeta_i}\left(t_i\overline{q}_i(t_i) - \int_0^{t_i}\overline{q}_i(t_i)dt_i\right) f_{i,\zeta_i}(t_i)dt_i,
\end{aligned}$$

which by the envelope-theorem routine is equal to the ex ante expected payment of type ζ_i in player i_*'s mechanism $\mathbb{M}(\zeta)$ conditional on the event that $\boldsymbol{t}_k \in [0, \zeta_k]$ for all $k \neq i_*$. Thus, by the definition of the payment rule in that mechanism,

$$Y = \mathbb{E}\left[\mathcal{V}_{i,\zeta_i}^{-1}\left(v_i(t_{i_*}, \boldsymbol{t}_{-(i,i_*)})\right)\mathbf{1}\left[v_i(t_{i_*}, \boldsymbol{t}_{-(i,i_*)}) < \mathcal{V}_{i,\zeta_i}(\boldsymbol{t}_i)\right] \mid \boldsymbol{t}_{-i_*} \leqq \zeta_{-i_*}\right].$$

This combined with Ineq. (54) gives the desired inequality:

$$\overline{R}(\zeta, t_{i_*}) < \mathbb{E}\left[\mathcal{V}_{i,\zeta_i}^{-1}\left(v_i(t_{i_*}, \boldsymbol{t}_{-(i,i_*)})\right) \mid \boldsymbol{t}_{-i_*} \leqq \zeta_{-i_*}\right] = \mathbb{E}\left[\mathcal{V}_{i,\zeta_i}^{-1}\left(v_i(t_{i_*}, \boldsymbol{t}_{-(i,i_*)})\right) \mid \boldsymbol{t}_{-(i,i_*)} \leqq \zeta_{-(i,i_*)}\right]. \blacksquare$$

A.1.2 Monotonicity of Optimal Expected Revenues

With monotone bidding strategies in the initial auction, a higher winning bid implies higher supremums of the posterior supports of the losing bidders. If the reseller adjusts her resale mechanism accordingly, the effect on her expected revenues is quantified below.

Proposition 4 *If $\zeta \leqq \zeta'$ and $\zeta \neq \zeta$, then $\overline{R}(\zeta, t_{i_*}) \leq \overline{R}(\zeta', t_{i_*})$ and:*

a. if $\max_{k\neq i_*}\zeta_k \le t_{i_*} < \max_{k\neq i_*}\zeta'_k$, *then*

$$\overline{R}(\zeta', t_{i_*}) - \overline{R}(\zeta, t_{i_*}) \geq \left(1 - \frac{F_k\left(\mathcal{V}^{-1}_{k,\zeta'_k}(t_{i_*})\right)}{F_k\left(\zeta'_k\right))}\right)\left(\mathcal{V}^{-1}_{k,\zeta'_k}(t_{i_*}) - t_{i_*}\right); \tag{55}$$

b. if $t_{i_*} < \max_{k\neq i_*}\zeta_k$ *then*

$$\overline{R}(\zeta', t_{i_*}) - \overline{R}(\zeta, t_{i_*}) \geq \gamma \Pr\left\{\exists k \in I\setminus\{i_*\} : \zeta_k < \boldsymbol{t}_k \le \zeta'_k]\right\}, \tag{56}$$

where

$$\gamma := \frac{\prod_{k\neq i} F_k\left(\zeta_k\right) - \prod_{k\neq i} F_k\left(\min\left\{\mathcal{V}^{-1}_{k,\zeta_k}(t_{i_*}), \zeta_k\right\}\right)}{\left(\prod_{k\neq i} F_k\left(\zeta'_k\right)\right)\left(\prod_{k\neq i} F_k\left(\zeta_k\right)\right)} t_{i_*}. \tag{57}$$

Proof There are only two possible cases: (i) $t_{i_*} \geq \max_{k\neq i_*}\zeta_k$ and (ii) $t_{i_*} < \max_{k\neq i_*}\zeta_k$.

Case (i): $t_{i_*} \geq \max_{k\neq i_*}\zeta_k$. Then there is no gain of trade and hence $\overline{R}(\zeta, t_{i_*}) = t_{i_*}$. By the fact that $\overline{R}(\zeta', t_{i_*}) \geq t_{i_*}$, we have $\overline{R}(\zeta', t_{i_*}) \geq \overline{R}(\zeta, t_{i_*})$. Now suppose, in addition, that $t_{i_*} < \zeta'_k$ for some $k \neq i_*$, which is the case in Claim (a) of the proposition. Then there is a strictly positive probability that sale happens at mechanism $\mathbb{M}(\zeta', t_{i_*})$; in the case of sale, seller i_*'s payoff is at least as large as the reserve price $\mathcal{V}^{-1}_{k,z_k}(t_{i_*})$. Hence Ineq. (55) follows.

Case (ii): $t_{i_*} < \max_{k\neq i_*}\zeta_k$. Note that the mechanism $\mathbb{M}(\zeta, t_{i_*})$ is ex post incentive feasible for any potential buyer $k \in I\setminus\{i_*\}$: Conditional on any $t_{-k} \in T_{-k}$, k's winning probability in $\mathbb{M}(\zeta', t_{i_*})$ is nondecreasing in t_k since k's virtual utility $\mathcal{V}_{k,\zeta_k}(t_k)$ is so, and k's payment $\mathcal{V}^{-1}_{k,\zeta_k}(v(t_{-k}))$, denoted $p_k(t_{-k})$ here, satisfies the envelope equation conditional on t_{-k}. Thus, when the support supremums are $(\zeta'_k)_{k\neq i_*}$ instead of $(\zeta_k)_{k\neq i_*}$, $\mathbb{M}(\zeta, t_{i_*})$ is still incentive feasible. Hence let $\widehat{R}(\zeta \mid \zeta')$ denote seller i_*'s expected payoff generated by $\mathbb{M}(\zeta, t_{i_*})$ at the truthtelling equilibrium given distributions $(F_{k,\zeta'_k})_{k\neq i_*}$. By revealed preference from i_*'s viewpoint,

$$\overline{R}(\zeta', t_{i_*}) - \overline{R}(\zeta, t_{i_*}) \geq \widehat{R}(\zeta \mid \zeta') - \overline{R}(\zeta, t_{i_*}). \tag{58}$$

Denote

$$\begin{aligned}
A &:= \left\{t_{-i} \in T_{-i} : t_{-i_*} \leqq \zeta; t_{i_*} > \max_{k\neq i_*}\mathcal{V}_{k,\zeta_k}(t_k)\right\},\\
B &:= \left\{t_{-i} \in T_{-i} : t_{-i_*} \nleqq \zeta; t_{-i_*} \leqq \zeta'; t_{i_*} > \max_{k\neq i}\mathcal{V}_{k,\zeta_k}(t_k)\right\},\\
C &:= \{t_{-i} \in T_{-i} : t_{-i_*} \leqq \zeta\}\setminus A,\\
D &:= \{t_{-i} \in T_{-i} : t_{-i_*} \nleqq \zeta; t_{-i_*} \leqq \zeta'\}\setminus B.
\end{aligned}$$

Thus, $A \cup C$ is the support of $\boldsymbol{t}_{-i_*}$ given ζ; within $A \cup C$, A is the event in which i_* does not sell the good at mechanism $\mathbb{M}(\zeta, t_{i_*})$. Analogously, $A \cup B \cup C \cup D$ is the support of $\boldsymbol{t}_{-i_*}$ given ζ', and $A \cup B$ the event of no-sale at mechanism $\mathbb{M}(\zeta, t_{i_*})$.

Let $\pi(A)$, $\pi(B)$, $\pi(C)$ and $\pi(D)$ denote the prior probabilities of these sets. Since $t_{i_*} < \zeta_k$ for some $k \neq i$ in this case, $\pi(C) > 0$ (since $\mathcal{V}_{k,\zeta_k} \leq \zeta_k$). Let us compare the performance of mechanism $\mathbb{M}(\zeta, t_{i_*})$ conditional on ζ with its performance conditional ζ':

	$A \cup B$	C	D
probability given ζ	$\frac{\pi(A)}{\pi(A)+\pi(C)}$	$\frac{\pi(C)}{\pi(A)+\pi(C)}$	0
probability given ζ'	$\frac{\pi(A)+\pi(B)}{\pi(A)+\pi(B)+\pi(C)+\pi(D)}$	$\frac{\pi(C)}{\pi(A)+\pi(B)+\pi(C)+\pi(D)}$	$\frac{\pi(D)}{\pi(A)+\pi(B)+\pi(C)+\pi(D)}$
ex post payoff for i	t_{i_*}	$\sum_{j \neq i_*} q_{ji_*}(t) p_j(t_{-j})$	$\sum_{j \neq i_*} q_{ji_*}(t) p_j(t_{-j})$

In the cells on the last row and the third and fourth columns, the ex post payoff for seller i_* is equal to $\sum_{j \neq i_*} q_{ji_*}(t) p_j(t_{-j})$, where $q_{ji_*}(t)$ denotes the probability with which i_* sells the good to j in the mechanism $\mathbb{M}(\zeta, t_{i_*})$. According to the payment rule in $\mathbb{M}(\zeta, t_{i_*})$, this sum of payments is at least as large as the reserve price $\mathcal{V}_{j,\zeta_j}^{-1}(t_{i_*})$ for any j who wins; this reserve price is strictly greater than t_{i_*} as $t_{i_*} < \zeta_j'$ in this case. Therefore,

$$
\begin{aligned}
& \widehat{R}(\zeta \mid \zeta') - \overline{R}(\zeta, t_{i_*}) \\
> \; & t_{i_*} \left(\frac{\pi(D)}{\pi(A)+\pi(B)+\pi(C)+\pi(D)} - 0 + \frac{\pi(A)+\pi(B)}{\pi(A)+\pi(B)+\pi(C)+\pi(D)} - \frac{\pi(A)}{\pi(A)+\pi(C)} \right) \\
= \; & \frac{\pi(C)\,(\pi(B)+\pi(D))}{(\pi(A)+\pi(B)+\pi(C)+\pi(D))\,(\pi(A)+\pi(C))} t_{i_*}.
\end{aligned}
$$

which is equal to $\pi(B) + \pi(D)$ multiplied by the γ defined by Eq. (57). Note that $\pi(B) + \pi(D) = \Pr\{\exists k \in I \setminus \{i_*\} : \zeta_k < \boldsymbol{t}_k \leq \zeta_k']\}$. Thus, Ineq. (58) implies (56). ■

A.2 Semi-Regular Resale Environments

A *semi-regular* resale environment is characterized by a vector $\zeta := (\zeta_k)_{k \in I \setminus \{i_*\}}$ and an $i \in I \setminus \{i_*\}$ as follows: the distribution of $\boldsymbol{t}_k$ for each $k \in I \setminus \{i, i_*\}$ is the same as in the regular environment characterized by ζ; for i, however, there is an $\alpha_i \in (0, \zeta_i]$ and a $\pi_i \in (0,1)$ such that the distribution of $\boldsymbol{t}_i$ is

$$
F_i(t_i \mid \alpha_i, \zeta_i) := \begin{cases} \frac{F_i(t)}{\pi_i F_i(\zeta_i) + (1-\pi_i) F_i(\alpha_i)} & \text{if } 0 \leq t_i \leq \alpha_i \\ \frac{\pi_i F_i(t) + (1-\pi_i) F_i(\alpha_i)}{\pi_i F_i(\zeta_i) + (1-\pi_i) F_i(\alpha_i)} & \text{if } \alpha_i \leq t_i \leq \zeta_i. \end{cases} \tag{59}
$$

Thus, while the posterior virtual utility function of each $k \in I \setminus \{i, i_*\}$ is the same as $\mathcal{V}_{k,\zeta_k}$ defined in the regular case, the counterpart for player i is the ironed version of—

$$\mathcal{V}_{i,\alpha_i,\zeta_i}(t_i) := \begin{cases} t_i - \frac{\pi_i F_i(\zeta_i)+(1-\pi_i)F_i(\alpha_i)-F_i(t_i)}{f_i(t_i)} & \text{if } 0 \leq t_i < \alpha_i \\ t_i - \frac{F_i(\zeta_i)-F_i(t_i)}{f_i(t_i)} & \text{if } \alpha_i < t_i \leq \zeta_i. \end{cases}$$

(Note that if $\alpha_i = \zeta_i$ then $F_i(\cdot \mid \alpha_i, \zeta_i)$ is specialized back to the regular case.) Since $t_i - (1 - F_i(t_i))/f_i(t_i)$ is strictly increasing in t_i on the prior support, one can show that $\mathcal{V}_{i,\alpha_i,\zeta_i}$ strictly increases on $[0, \alpha_i)$, drops at the point α_i, and then strictly increases on $(\alpha_i, \zeta_i]$, attaining to its maximum value ζ_i at ζ_i. Let $\overline{\mathcal{V}}_{i,\alpha_i,\zeta_i}$ denote the ironed version of $\mathcal{V}_{k,\zeta_k}$ according to Myerson's procedure. Denote $\mathbb{M}(\zeta, \alpha_i, t_{i_*})$ for the Myerson auction in this environment, given the seller's realized type t_{i_*}, and $R(\zeta, \alpha_i, t_{i_*}, t_{-i_*})$ the ex post payoff for the seller generated by this mechanism when t_{-i_*} is the realized type profile across all potential buyers. Note from the definition of Myerson's auction that, if i wins given profile t_{-i} of realized types across other players, then i's payment is equal to

$$\overline{\mathcal{V}}^{-1}_{i,\alpha_i,\zeta_i}(v_i(t_{-i})) := \inf \left\{ t_i \in [0, \zeta_i] : \overline{\mathcal{V}}_{i,\alpha_i,\zeta_i}(t_i) \geq v_i(t_{-i}) \right\}.$$

If the mechanism is given to be $\mathbb{M}(\zeta, \alpha_i, t_{i_*})$, how does a perturbation of bidder i's type distribution affect the seller's expected revenue? Regarding this question two properties are proved next, where the symbols ζ, α_i and t_{i_*} are suppressed (e.g., $\mathcal{V}_k$ means $\mathcal{V}_{k,\zeta_k}$) unless clarification is necessary.

A.2.1 Continuity

Given ζ, α_i and t_{i_*}, for any $S \subseteq \prod_{k \notin \{i,i_*\}}[0, \zeta_k]$ and $\Pr(S) > 0$ and any $t_i \in T_i$, define

$$\phi(t_i, S) := \mathbb{E}\left[R\left(\zeta, \alpha_i, t_i, t_{i_*}, \boldsymbol{t}_{-(i,i_*)}\right) \mid \boldsymbol{t}_{-(i,i_*)} \in S \right]. \tag{60}$$

Lemma 18 *If $S \subseteq \prod_{k \notin \{i,i_*\}}[0, \zeta_k]$ and $\Pr(S) > 0$, then $\phi(\cdot, S)$ is continuous on $[0, \zeta_i]$.*

Proof Let $\zeta_i \geq x'' > x' \geq 0$. Suppose the value of t_i increases from x' to x''. Given any t_{-i}, this change affects the seller's ex post payoff $R(\zeta, \alpha_i, t_i, t_{i_*}, t_{-(i,i_*)})$ in only two cases:

Case (i): $\overline{\mathcal{V}}_i(x') < v_i(t_{-i}) \leq \overline{\mathcal{V}}_i(x'')$. Let k_* be the bidder whose virtual utility is the highest when $t_i = x'$. Then, when t_i increases from x' to x'', the winner in $\mathbb{M}(\zeta, \alpha_i, t_{i_*})$

switches from k_* to i and the revenue for i_* changes by an amount bounded by $\max_{k\in I}\bar{t}_k$. Note that Case (i) occurs only if $\overline{\mathcal{V}}_i(x') < v_i(t_{-i}) \le \overline{\mathcal{V}}_i(x'')$, which belongs to the event

$$\bigcup_{j\notin\{i,i_*\}}\left\{t_{-(i,i_*)} \in \prod_{k\notin\{i,i_*\}}[0,\zeta_k] : \overline{\mathcal{V}}_i(x') < \max_{k\notin\{i,i_*\}}\mathcal{V}_k(t_k) = \mathcal{V}_j(t_j) \le \overline{\mathcal{V}}_i(x'')\right\}.$$

Case (ii): $\overline{\mathcal{V}}_i(x'') < v_i(t_{-i})$ and $\overline{\mathcal{V}}_i(x'')$ is equal to the second highest among $(\mathcal{V}_k(t_k))_{k\notin\{i,i_*\}}$ and $\overline{\mathcal{V}}_i(x'')$. Let k^* be the bidder whose virtual utility is the highest when $t_i = x''$ (hence also the highest when $t_i = x'$). When t_i increases from x' to x'', k^* remains to be the winner in $\mathbb{M}(\zeta, \alpha_i, t_{i_*})$ but her payment according to Eq. (51) increases by the amount

$$\mathcal{V}_{k^*}^{-1}\left(\overline{\mathcal{V}}_i(x'')\right) - \mathcal{V}_{k^*}^{-1}\left(\max\left\{t_{i_*}, \overline{\mathcal{V}}_i(x'), \max_{k\notin\{i,i_*k^*\}}\mathcal{V}_k(t_k)\right\}\right) \;\le\; \mathcal{V}_{k^*}^{-1}\left(\overline{\mathcal{V}}_i(x'')\right) - \mathcal{V}_{k^*}^{-1}\left(\overline{\mathcal{V}}_i(x')\right),$$

where $\mathcal{V}_{k^*}^{-1}$ exists and is continuous as $\mathcal{V}_{k^*}$ is strictly increasing and continuous on $[0, \zeta_{k^*}]$.

Thus, in either case, the difference is in the order of $\left|\mathcal{V}_{k^*}^{-1}\left(\overline{\mathcal{V}}_i(x'')\right) - \mathcal{V}_{k^*}^{-1}\left(\overline{\mathcal{V}}_i(x')\right)\right|$, which is in the order of $|x''-x'|$ due to the continuity of $\overline{\mathcal{V}}_i$ and $\mathcal{V}_{k^*}^{-1}$. Thus, $|\phi(x'', S) - \phi(x', S)|$ is in the order of $|x'' - x'|$. ■

Given $(\zeta^n, S^n)_{n=1}^\infty$ such that $S^n \subseteq \prod_{k\notin\{i,i_*\}}[0,\zeta_k^n]$ and $\Pr(S^n) > 0$ for each n, define

$$\phi^n(t_i) := \mathbb{E}\left[R\left(\zeta^n, \zeta_i^n, t_i, t_{i_*}, \boldsymbol{t}_{-(i,i_*)}\right) \mid \boldsymbol{t}_{-(i,i_*)} \in S^n\right].$$

Corollary 2 *If $\zeta^n \to \zeta$ and $S^n \to S$, then $(\phi^n)_{n=1}^\infty$ is equicontinuous at ζ_i.*

Proof Denote the posterior virtual utilities in the environment ζ^n by $\mathcal{V}_k^n$ ($k \notin \{i, i_*\}$) and $\overline{\mathcal{V}}_i^n$. By the proof of Lemma 18, there exists $K > 0$ such that for any n and any $t_i \in T_i$

$$|\Pr(S^n)\left(\phi^n(t_i) - \phi^n(\zeta_i)\right)| \le K\left|(\mathcal{V}_{k^*}^n)^{-1}\left(\overline{\mathcal{V}}_i^n(t_i)\right) - (\mathcal{V}_{k^*}^n)^{-1}\left(\overline{\mathcal{V}}_i^n(\zeta_i)\right)\right| \le (K/\lambda)\left|\overline{\mathcal{V}}_i^n(t_i) - \overline{\mathcal{V}}_i^n(\zeta_i)\right|$$

for some $k^* \notin \{i, i_*\}$, where the second inequality follows from the fact that the derivative of the increasing inverse function $(\mathcal{V}_{k^*}^n)^{-1}$ is bounded from above by $1/\lambda$ (Lemma 2.a.ii). By the definition of virtual utilities and the ironing procedure, $\overline{\mathcal{V}}_i^n(t_i) = \mathcal{V}_i^n(t_i)$ when $t_i \uparrow \zeta_i^n$ and $\overline{\mathcal{V}}_i^n(t_i) = \zeta_i^n$ when $t_i \ge \zeta_i^n$. Specifically, the right-derivative of $\overline{\mathcal{V}}_i^n$ at ζ_i^n is equal to zero and the left-derivative of $\overline{\mathcal{V}}_i^n$ at ζ_i^n is

$$D_-\overline{\mathcal{V}}_i^n(\zeta_i^n) = D_-\mathcal{V}_i^n(\zeta_i^n) = \lim_{t_i\uparrow\zeta_i^n}\left(\zeta_i^n - t_i + \frac{F_i(\zeta_i^n) - F_i(t_i)}{f_i(t_i)}\right)\frac{1}{\zeta_i^n - t_i} = 2.$$

Thus, with $\zeta_i^n \to_n \zeta$, the left- and right-derivatives of $\overline{\mathcal{V}}_i^n$ at ζ_i are bounded between zero and 2. This being true for all n, we have shown that $(\phi^n)_{n=1}^\infty$ is equicontinuous at ζ_i. ■

A.2.2 Monotonicity with Respect to Upward Weight Transfers

Denote, for any $\theta \in [0, \zeta_i]$ and any $t_i \in T_i$,

$$\varphi_i(t_i, t_{i_*}) := \mathbb{E}\left[R\left(\zeta, \alpha_i, t_{i_*}, t_i, \boldsymbol{t}_{-(i,i_*)}\right) \mid \boldsymbol{t}_{-(i,i_*)} \leqq \zeta_{-(i,i_*)}\right], \tag{61}$$

$$\Phi(\theta, t_{i_*}) := \frac{1}{1 - F_i(\theta \mid \alpha_i, \zeta_i)} \int_\theta^{\zeta_i} \varphi_i(\tau_i, t_{i_*}) F_i(d\tau_i \mid \alpha_i, \zeta_i). \tag{62}$$

Note that $\Phi(0, t_{i_*})$ is equal to i_*'s expected payoff generated by $\mathbb{M}(\zeta, \alpha_i, t_{i_*})$ given posterior distributions $\left((F_{k,\zeta_k})_{k \notin \{i,i_*\}}, F_i(\cdot \mid \alpha_i, \zeta_i)\right)$ and, given these distributions, is the maximum expected payoff for the type-t_{i_*} seller i_* among all incentive feasible game forms for $k \neq i_*$.

Lemma 19 *For any $t_{i_*} \in T_{i_*}$, any $i \neq i_*$ and any $\theta \in [0, \zeta_i)$, $\Phi(\theta, t_{i_*}) \geq \Phi(0, t_{i_*})$.*

Proof Suppose, to the contrary, that $\Phi(\theta, t_{i_*}) < \Phi(0, t_{i_*})$ for some $\theta \in (0, \zeta_i)$. Let $S := \{t_i \in [\theta, \zeta_i] : \varphi_i(t_i, t_{i_*}) < \Phi(0, t_{i_*})\}$. Then $\Pr(S) > 0$, so $\inf S < \zeta_i$. Let

$$x := \begin{cases} \inf S & \text{if } \inf S > \theta \\ \sup\{t_i \in [0, \theta] : \varphi_i(t_i, t_{i_*}) \geq \Phi(0, t_{i_*})\} & \text{if } \inf S = \theta. \end{cases}$$

Then $x < t_i < \theta \Rightarrow \varphi_i(t_i, t_{i_*}) < \Phi(0, t_{i_*})$, and $\theta < t_i < x \Rightarrow \varphi_i(t_i, t_{i_*}) \geq \Phi(0, t_{i_*})$. Thus, $\Phi(x, t_{i_*}) < \Phi(0, t_{i_*})$ and $x > 0$. By continuity of $\varphi_i(\cdot, t_{i_*})$ (Lemma 18), $\varphi_i(x, t_{i_*}) = \Phi(0, t_{i_*})$.

Now consider a mechanism $\widehat{M}$ which is the same as $\mathbb{M}(\zeta, \alpha_i, t_{i_*})$ except that bidder i's virtual utility function $\overline{\mathcal{V}}_i$ is replaced by a function $\widehat{V}_i$ defined by

$$\widehat{V}_i(t_i) := \begin{cases} \overline{\mathcal{V}}_i(t_i) & \text{if } t_i \leq x \\ \overline{\mathcal{V}}_i(x) & \text{if } t_i \geq x. \end{cases}$$

$\widehat{M}$ is incentive compatible: $\widehat{V}_i$ is nondecreasing and hence bidder i's probability of winning is nondecreasing in his type; the monotonicity of the other bidders' winning probabilities is unaffected. The payment rule satisfies the envelope formula because the payment is defined according to the formula based on $\left(\widehat{V}_i, (\mathcal{V}_k)_{k \notin \{i,i_*\}}\right)$. Individual rationality of $\widehat{M}$ is obvious.

Thus, we may assume that bidders participate and are truthful in $\widehat{M}$. When bidder i's type is any $t_i \in [0, x]$, $\widehat{M}$ acts in the same way as $\mathbb{M}(\zeta, \alpha_i, t_{i_*})$, generating the same expected revenue $\varphi_i(t_i, t_{i_*})$ conditional on t_i. When $t_i > x$, by contrast, $\widehat{M}$ acts as $\mathbb{M}(\zeta, \alpha_i, t_{i_*})$ except that t_i is turned into x, so the expected revenue conditional on t_i becomes $\varphi_i(x, t_{i_*})$.

Thus, the expected revenue generated by $\widehat{M}$ is equal to a convex combination between $\frac{1}{F_i(x|\alpha_i,\zeta_i)} \int_0^x \varphi_i(t_i, t_{i_*}) F_i(dt_i \mid \alpha_i, \zeta_i)$ and $\varphi_i(x, t_{i_*})$. As noted above, $\varphi_i(x, t_{i_*}) = \Phi(0, t_{i_*})$; also, by the fact $\Phi(x, t_{i_*}) < \Phi(0, t_{i_*})$ proved above,

$$\frac{1}{F_i(x \mid \alpha_i, \zeta_i)} \int_0^x \varphi_i(t_i, t_{i_*}) F_i(dt_i \mid \alpha_i, \zeta_i) > \Phi(0, t_{i_*}).$$

Thus, with the fact $x > 0$, the expected revenue yielded by $\widehat{M}$ is greater than $\Phi(0, t_{i_*})$, which contradicts the fact that $\Phi(0, t_{i_*})$ is maximum among all incentive feasible mechanisms. ■

Corollary 3 *For any measurable $S \subseteq T_{i_*}$, $\mathbb{E}\left[\varphi_i(\cdot, \boldsymbol{t}_{i_*}) \mathbf{1}_{\boldsymbol{t}_{i_*} \in S}\right]$ is continuous on $[0, \zeta_i]$, and*

$$\forall \theta \in (0, \zeta_i) : \mathbb{E}\left[\Phi(\theta, \boldsymbol{t}_{i_*}) \mathbf{1}_{\boldsymbol{t}_{i_*} \in S}\right] \geq \mathbb{E}\left[\Phi(0, \boldsymbol{t}_{i_*}) \mathbf{1}_{\boldsymbol{t}_{i_*} \in S}\right]. \tag{63}$$

Proof By Lemma 18, $\varphi(\cdot, t_{i_*})$ is continuous; by Lemma 19, $\Phi(\theta, t_{i_*}) \geq \Phi(0, t_{i_*})$. Thus, integration of $\varphi(\cdot, t_{i_*})$ across all $t_{i_*} \in S$ preserves the continuity, and integration of $\Phi(\theta, t_{i_*}) \geq \Phi(0, t_{i_*})$ across all $t_{i_*} \in S$ gives Ineq. (63). ■

Corollary 4 *If the distribution of $\boldsymbol{t}_i$ changes from the semi-regular $F_i(\cdot \mid \alpha_i, \zeta_i)$ to the regular F_{i,ζ_i}, then seller i_*'s expected payoff generated by $\mathbb{M}(\zeta, \alpha_i, t_{i_*})$ cannot decrease:*

$$\int_0^{\zeta_i} \varphi_i(t_i, t_{i_*}) F_{i,\zeta_i}(dt_i) \geq \Phi(0, t_{i_*}).$$

Proof To avoid triviality, let $\alpha_i < \zeta_i$. Denote

$$X := \frac{1}{F_i(\alpha_i \mid \alpha_i, \zeta_i)} \int_0^{\alpha_i} \varphi_i(t_i, t_{i_*}) F_i(dt_i \mid \alpha_i, \zeta_i).$$

By Eq. (62),

$$\Phi(0, t_{i_*}) = F_i(\alpha_i \mid \alpha_i, \zeta_i) X + (1 - F_i(\alpha_i \mid \alpha_i, \zeta_i)) \Phi(\alpha_i, t_{i_*}). \tag{64}$$

By Eqs. (59) and (62),

$$\begin{aligned} X &= \frac{1}{F_i(\alpha_i)} \int_0^{\alpha_i} \varphi_i(t_i, t_{i_*}) F_i(dt_i), \\ \Phi(\alpha_i, t_{i_*}) &= \frac{1}{F_i(\zeta_i) - F_i(\alpha_i)} \int_0^{\alpha_i} \varphi_i(t_i, t_{i_*}) F_i(dt_i). \end{aligned}$$

Combined with Eq. (50), these two equations together imply

$$\int_0^{\zeta_i} \varphi_i(t_i, t_{i_*}) F_{i,\zeta_i}(dt_i) = \frac{1}{F_i(\zeta_i)} \int_0^{\zeta_i} \varphi_i(t_i, t_{i_*}) F_i(dt_i) = \frac{F_i(\alpha_i)}{F_i(\zeta_i)} X + \left(1 - \frac{F_i(\alpha_i)}{F_i(\zeta_i)}\right) \Phi(\alpha_i, t_{i_*}). \tag{65}$$

Since $\alpha_i < \zeta_i$, Lemma 19 implies $\Phi(\alpha_i, t_{i_*}) \geq \Phi(0, t_{i_*})$; hence Eq. (64) implies $X \leq \Phi(\alpha_i, t_{i_*})$. Thus, with $F_i(\alpha_i \mid \alpha_i, \zeta_i) \geq F_i(\alpha_i)/F_i(\zeta_i)$ and $1 - F_i(\alpha_i \mid \alpha_i, \zeta_i) \leq 1 - F_i(\alpha_i)/F_i(\zeta_i)$, Eqs. (64) and (65) together imply that $\Phi(0, t_{i_*}) \leq \int_0^{\zeta_i} \varphi_i(t_i, t_{i_*}) F_{i,\zeta_i}(dt_i)$, as desired. ■

B A Semicontinuity Property of Monotone Functions

For any weakly increasing function $g : [a, z] \to \mathbb{R}$, define $g_{\inf}^{-1}(y)$ and $g_{\sup}^{-1}(y)$ by Eqs. (1) and (2), with g taking the role of β_k.

Lemma 20 *Let $g : [a, z] \to \mathbb{R}$ be a weakly increasing function with $a < z$. For any $y \geq g(a)$ and any $\epsilon > 0$, there exists $\delta > 0$ such that if $y - \delta < y' < y + \delta$ then*

$$g_{\inf}^{-1}(y) - \epsilon < g_{\inf}^{-1}(y') \leq g_{\sup}^{-1}(y') < g_{\sup}^{-1}(y) + \epsilon. \tag{66}$$

Proof First, suppose $g(a) < y < g(z)$. Let $\epsilon > 0$. As g is weakly increasing, we can shrink ϵ so that $g_{\inf}^{-1}(y) - \frac{\epsilon}{2}$ and $g_{\sup}^{-1}(y) + \frac{\epsilon}{2}$ each belong to $[a, z]$. By definitions of $g_{\inf}^{-1}(y)$ and $g_{\sup}^{-1}(y)$,

$$g\left(g_{\inf}^{-1}(y) - \frac{\epsilon}{2}\right) < y < g\left(g_{\sup}^{-1}(y) + \frac{\epsilon}{2}\right). \tag{67}$$

Let

$$\delta := \min\left\{\frac{g\left(g_{\sup}^{-1}(y) + \frac{\epsilon}{2}\right) + y}{2} - y, y - \frac{g\left(g_{\inf}^{-1}(y) - \frac{\epsilon}{2}\right) + y}{2}\right\}.$$

Then $\delta > 0$. Pick any y' such that $y - \delta < y' < y + \delta$. Either (i) $g^{-1}(y') = \varnothing$ or (ii) $g^{-1}(y') \neq \varnothing$. In case (i), since

$$g\left(g_{\inf}^{-1}(y) - \epsilon/2\right) < \frac{g\left(g_{\inf}^{-1}(y) - \epsilon/2\right) + y}{2} \leq y - \delta < y',$$

by Eq. (1) we have $g_{\inf}^{-1}(y') \geq g_{\inf}^{-1}(y) - \epsilon/2 > g_{\inf}^{-1}(y) - \epsilon$. In case (ii), if $x \in g^{-1}(y)$ and $x < g_{\inf}^{-1}(y) - \epsilon/2$, then monotonicity of g implies

$$y' = g(x) \leq g\left(g_{\inf}^{-1}(y) - \epsilon/2\right) < \frac{g\left(g_{\inf}^{-1}(y) - \epsilon/2\right) + y}{2} \leq y - \delta,$$

contradicting the fact that $y' > y - \delta$; thus, $g_{\inf}^{-1}(y') \geq g_{\inf}^{-1}(y) - \epsilon/2 > g_{\inf}^{-1}(y) - \epsilon$. Analogously, we can show $g_{\sup}^{-1}(y') < g_{\sup}^{-1}(y) + \epsilon$. Thus, (66) holds if $g(a) < y < g(z)$.

Next consider the case where $y \geq g(z)$. If $y > g(z)$ then, for any sufficiently small $\delta > 0$, $y - \delta > g(z)$ and hence $y - \delta < y' < y + \delta$ implies $g_{\inf}^{-1}(y') = g_{\sup}^{-1}(y') = z = g_{\inf}^{-1}(y) = g_{\sup}^{-1}(y)$. If $y = g(z)$ then we just replace the upper bound $g\left(g_{\sup}^{-1}(y) + \frac{\epsilon}{2}\right)$ in Ineq. (67) by any number bigger than y, and then the calculation in the previous paragraph follows. The case where $y \leq g(a)$ is analogous. ■

C Posterior Densities and Virtual Utilities

Proof of Lemma 1 Denote $\pi_k(i,b_i)$ for the probability of the event that bidder k, conditional on submitting the highest bid b_i (thereby tying with i and possibly others), loses the tie-breaking lottery. Then (with the convention of letting $\prod_{j\in\varnothing} x_j := 1$)

$$\pi_k(i,b_i) = \sum_{S\subseteq I\setminus\{i,k\}} \frac{|S|+1}{|S|+2}\left(\prod_{j\in S}(F_j(\beta^{-1}_{j,\sup}(b_i)) - F_j(\beta^{-1}_{j,\inf}(b_i)))\right)\left(\prod_{j\in I\setminus(S\cup\{i,k\})} F_j(\beta^{-1}_{j,\inf}(b_i))\right).$$

Note that $\pi_k(i,b_i)$ is independent of t_k. By Bayes's rule,

$$F_k(t_k \mid i,b_i,\beta) = \begin{cases} \frac{F_k(t_k)}{F_k(\beta^{-1}_{k,\inf}(b_i))+(F_k(\beta^{-1}_{k,\sup}(b_i))-F_k(\beta^{-1}_{k,\inf}(b_i)))\pi_k(i,b_i)} & \text{if } t_k \le \beta^{-1}_{k,\inf}(b_i) \\ \frac{F_k(\beta^{-1}_{k,\inf}(b_i))+(F_k(t_k)-F_k(\beta^{-1}_{k,\inf}(b_i)))\pi_k(i,b_i)}{F_k(\beta^{-1}_{k,\inf}(b_i))+(F_k(\beta^{-1}_{k,\sup}(b_i))-F_k(\beta^{-1}_{k,\inf}(b_i)))\pi_k(i,b_i)} & \text{if } \beta^{-1}_{k,\inf}(b_i) \le t_k \le \beta^{-1}_{k,\sup}(b_i). \end{cases} \tag{68}$$

Thus, the density

$$f_k(t_k \mid i,b_i,\beta) = \begin{cases} \frac{f_k(t_k)}{F_k(\beta^{-1}_{k,\inf}(b_i))+(F_k(\beta^{-1}_{k,\sup}(b_i))-F_k(\beta^{-1}_{k,\inf}(b_i)))\pi_k(i,b_i)} & \text{if } t_k < \beta^{-1}_{k,\inf}(b_i) \\ \frac{f_k(t_k)\pi_k(i,b_i)}{F_k(\beta^{-1}_{k,\inf}(b_i))+(F_k(\beta^{-1}_{k,\sup}(b_i))-F_k(\beta^{-1}_{k,\inf}(b_i)))\pi_k(i,b_i)} & \text{if } \beta^{-1}_{k,\inf}(b_i) < t_k \le \beta^{-1}_{k,\sup}(b_i); \end{cases} \tag{69}$$

at the point $\beta^{-1}_{k,\inf}(b_i)$, the right density is equal to $\pi_k(i,b_i)$ times the left density. Thus, $f_k(\cdot \mid i,b_i,\beta)$ exists and is strictly positive on the posterior support and is continuous unless $\beta^{-1}_{k,\inf}(b_i) \ne \beta^{-1}_{k,\sup}(b_i)$, in which case $\beta^{-1}_{k,\inf}(b_i)$ is the only discontinuity point. ■

Proof of Lemma 2 For brevity, suppress the symbols i and β unless needed for clarification. Let $b' > b \ge r$ and consider any bidder k. Denote $y := \beta^{-1}_{k,\inf}(b)$ and $z := \beta^{-1}_{k,\sup}(b)$, denote y' and z' with respect to b' analogously. For any $t_k \le z$, by Eqs. (3), (68) and (69),

$$V_k(t_k \mid i,b) = \begin{cases} t_k - \frac{\pi_k(i,b)F_k(z)+(1-\pi_k(i,b))F_k(y)-F_k(t_k)}{f_k(t_k)} & \text{if } t_k < y \\ t_k - \frac{F_k(z)-F_k(t_k)}{f_k(t_k)} & \text{if } y < t_k \le z, \end{cases} \tag{70}$$

and $V_k(t_k \mid i',b')$ is analogous.

To prove Claim (a), suppose that b is not an atom of β_k. Then $y=z$, hence Eq. (70) implies Eq. (4). For any $t_k \in \left(0, \beta^{-1}_{k,\sup}(b_i)\right)$, the derivative of $V_{k,b_i,\beta}$, or $\frac{d}{dt_i}\left(t_i - \frac{F_k(\beta^{-1}_{k,\sup}(b_i))-F_k(t_k)}{f_k(t_k)}\right)$ by Eq. (4), is no less than either 2 or $\frac{d}{dt_i}\left(t_i - \frac{1-F_k(t_k)}{f_k(t_k)}\right)$, which, strictly positive on the compact T_i by assumption, is bigger than a $\lambda_k > 0$ constant to t_i. Thus, Claim (a.ii) follows, with $\lambda := \max_{k\in I}\min\{\lambda_k, 2\}$. Now that $V_{k,i,b,\beta}$ is nondecreasing on T_k (Claim (a.ii coupled with the lower branch of Eqs. (3)), the ironing procedure is unnecessary and hence Claim (a.i) follows. Then Claim (a.iii) follows trivially.

Next we prove Claim (b). Since $F_k(z) \geq F_k(y)$, Eq. (70) implies, with $t_k \leq z$,

$$V_k(t_k \mid i, b) \geq t_k - \frac{F_k(z) - F_k(t_k)}{f_k(t_k)}.$$

With $b' > b$ and β_k nondecreasing, we have $z \leq y' \leq z'$ and hence $F_k(z) \leq \pi_k(b')F_k(z') + (1 - \pi_k(b'))F_k(y')$. Thus,

$$t_k - \frac{F_k(z) - F_k(t_k)}{f_k(t_k)} \geq t_k - \frac{\pi_k(i', b')F_k(z') + (1 - \pi_k(i', b'))F_k(y') - F_k(t_k)}{f_k(t_k)} = V_k(t_k \mid i', b'),$$

where the equality holds because $t_k \leq z \leq y'$. Thus,

$$\forall t_k \in [0, z] : V_k(t_k \mid i, b) \geq V_k(t_k \mid i', b'). \tag{71}$$

When b is an atom of β_k, $y \neq z$ and the two branches of Eq. (70) do not coincide. Furthermore, as in the previous paragraph, the strict monotonicity assumption of the prior virtual utility implies that each branch is strictly increasing in t_k. Thus, $V_k(\cdot \mid i, b)$ strictly increases on $[0, y)$, drops at the point y, and then strictly increases again on $(y, z]$ (and reaches z when $t_k = z$). Analogous observation holds for $V_k(\cdot \mid i', b')$. By monotonicity of β_k, $y \leq z \leq y' \leq z'$.

Thus, based on the ironing procedure, $\overline{V}_k(t_k \mid i, b) < V_k(t_k \mid i, b)$ only if $t_k \leq y < z$, and $\overline{V}_k(t_k \mid i', b') > V_k(t_k \mid i', b')$ only if $y' \leq t_k < z'$. (See Figure 3.)

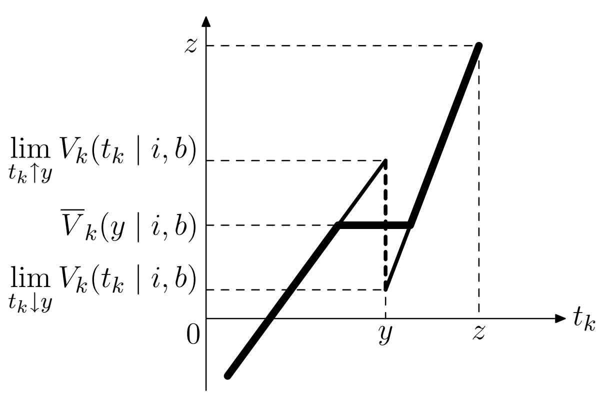

Figure 3: The thick curve: $\overline{V}_k(\cdot \mid i, b)$, the ironed posterior virtual utility

To extend Ineq. (71) to $\overline{V}$ thereby proving Claim (b), it suffices to consider the case where $\overline{V}_k(t_k \mid i, b) < V_k(t_k \mid i, b)$ or $\overline{V}_k(t_k \mid i', b') > V_k(t_k \mid i', b')$ is true. First, consider those t_k such that $\overline{V}_k(t_k \mid i, b) < V_k(t_k \mid i, b)$, which requires $t_k \leq y < z$. Now that $t_k \leq y < z \leq y'$,

$\overline{V}_k(t_k \mid i', b') \le V_k(t_k \mid i', b')$. By the ironing procedure, $\overline{V}_k(t_k \mid i, b) > \lim_{\tau \downarrow y} V_k(\tau \mid i, b)$. Since $y < z \le y'$, $V_k(\cdot \mid i', b')$ is increasing on $[0, z]$; thus, with $t_k \le y$,

$$V_k(t_k \mid i', b') \le \lim_{\tau \downarrow y} V_k(\tau \mid i', b') \overset{(71)}{\le} \lim_{\tau \downarrow y} V_k(\tau \mid i, b).$$

Thus,

$$\overline{V}_k(t_k \mid i', b') \le V_k(t_k \mid i', b') \le \lim_{\tau \downarrow y} V_k(\tau \mid i, b) < \overline{V}_k(t_k \mid i, b).$$

Second, consider those t_k such that $\overline{V}_k(t_k \mid i', b') > V_k(t_k \mid i', b')$, which requires $y' < z'$ and $t_k \ge y'$. By the ironing procedure, $\overline{V}_k(y' \mid i', b') < \lim_{\tau \uparrow y'} V_k(\tau \mid i', b')$. As the lemma covers only those t_k in $[0, z]$, with $z \le y'$ we need only to consider the case $z = y'$ and prove $\overline{V}(z \mid i, b) \ge \overline{V}(z \mid i', b')$. By the ironing procedure, $\overline{V}_k(z \mid i, b) = V_k(z \mid i, b) = z$ while $\lim_{\tau \uparrow z} V_k(\tau \mid i', b') < z$ by Eq. (3) and $z < z'$. Thus, $\overline{V}_k(z \mid i', b') = \overline{V}_k(y' \mid i', b') < \lim_{\tau \uparrow y'} V_k(\tau \mid i', b') < \overline{V}_k(z \mid i, b)$. Both cases considered, we have proved the lemma. ■

D Details of the Increasing Difference Theorem

Lemma 5 From player i's viewpoint, the continuation equilibrium is equivalent to an incentive feasible direct revelation mechanism that solicits from i an alleged type $\hat{t}_i$ and then plays the continuation equilibrium on his behalf in every possible event of the auction. In the event where i wins with bid b_i, i's period-two expected payoff is equal to $t_i \overline{Q}_i(b_i, \hat{t}_i, \beta) + R_i(b_i, \hat{t}_i, \beta)$, with t_i being his true type, $\overline{Q}_i(b_i, \hat{t}_i, \beta)$ his expected probability of being the final owner, and $R_i(b_i, \hat{t}_i, \beta)$ his expected revenue. The best-reply condition of the continuation equilibrium implies that the expected payoff is maximized when $\hat{t}_i = t_i$. Then the envelope theorem of Milgrom and Segal [11, Theorem 2], applicable because $\overline{Q}_i$ the partial derivative of this expected payoff with respect to t_i is uniformly bounded, implies (16). Analogously, the continuation equilibrium in the event that i does not win the period-one auction implies

$$\overline{L}_i(b_i, t_i, \beta) = \overline{L}_i(b_i, 0, \beta) + \int_0^{t_i} \bar{q}_i(b_i, \tau_i, \beta) d\tau_i.$$

The resale mechanism, optimal for the reseller, leaves zero surplus to the zero type of any other bidder. Thus, $\overline{L}_i(b_i, 0, \beta) = 0$ and Eq. (17) follows. ■

Propositions 1 Pick any $t := (t_i, t_{-i})$ such that $\max_{k \ne i} \beta_k(t_k) \le b'_i$. To avoid triviality, assume $b'_i \ge r$. By Lemma 2.b and $b''_i > b'_i$, $\overline{V}_k(t_k \mid i, b'_i, \beta) \ge \overline{V}_k(t_k \mid i, b''_i, \beta)$ for all

$k \neq i$. Thus, by provision (b) in definition of the resale mechanism (§3.1.4), if $Q_i(b_i', t, \beta) = 1$ then $t_i > \max_{k\neq i} \overline{V}_k(t_k \mid i, b_i', \beta) \geq \max_{k\neq i} \geq \overline{V}_k(t_k \mid i, b_i'', \beta)$, hence $Q_i(b_i'', t, \beta) = 1$. If $Q_i(b_i'', t, \beta) < 1$ then $t_i \leq \max_{k\neq i} \overline{V}_k(t_k \mid i, b_i'', \beta) \leq \max_{k\neq i} \overline{V}_k(t_k \mid i, b_i', \beta)$. Then either (i) at least one of the weak inequalities is strict or (ii) both weak inequalities are equalities. In case (i), $Q_i(b_i', t, \beta) = 0$. In case (ii), the final owner in either history, (i, b_i') and (i, b_i''), is randomly chosen with equal probability between i and those whose posterior virtual utilities are highest among $k \neq i$; since $\overline{V}_k(t_k \mid i, b_i', \beta) \geq \overline{V}_k(t_k \mid i, b_i'', \beta)$, there cannot be more of such finalists in the history (i, b_i'') than in (i, b_i'), hence $Q_i(b_i'', t, \beta) \geq Q_i(b_i', t, \beta)$. ■

Proposition 2 Pick any (t_i, t_{-i}) such that $b_i \geq \max_{k\neq i} \beta_k(t_k)$. To avoid triviality, assume $q_i(t, \beta) > 0$. That mans (i) some bidder $j \neq i$ wins, i.e., $\beta_j(t_j) = \max_{k\neq i} \beta_k(t_k) \geq r$, and (ii) j may resell the good to i, i.e., by provision (b) of the resale mechanism (§3.1.4),

$$\overline{V}_i(t_i \mid j, \beta_j(t_j), \beta) \geq \max \left\{ t_j, \max_{k \in I \setminus \{i,j\}} \overline{V}_k(t_k \mid j, \beta_j(t_j), \beta) \right\}. \tag{72}$$

By Lemma 2.b and $b_i \geq \beta_j(t_j) \geq \beta_k(t_k)$ for all $k \in I \setminus \{i, j\}$, we have $\overline{V}_k(t_k \mid j, \beta_j(t_j), \beta) \geq \overline{V}_k(t_k \mid i, b_i, \beta)$ for all $k \in I \setminus \{i, j\}$. Combining this with the fact that $t_j \geq \overline{V}_j(t_j \mid i, b_i, \beta)$ and $t_i \geq \overline{V}_i(t_i \mid j, \beta_j(t_j), \beta)$, each due to the definition of virtual utilities, we have $t_i \geq \max_{k\neq i} \overline{V}_k(t_k \mid i, b_i, \beta)$. If $t_i > \max_{k\neq i} \overline{V}_k(t_k \mid i, b_i, \beta)$ then $Q_i(b_i, t, \beta) = 1$. Otherwise, $t_i = \max_{k\neq i} \overline{V}_k(t_k \mid i, b_i, \beta)$, which forces (72) into an equality; then by provision (b) of the reale mechanism, in both histories (i, b_i) and $(j, \beta_j(t_j))$, the final owner is chosen via equal-probability lotteries from i and the other highest contenders. Since $\overline{V}_k(t_k \mid j, \beta_j(t_j), \beta) \geq \overline{V}_k(t_k \mid i, b_i, \beta)$ and $t_j \geq \overline{V}_j(t_j \mid i, b_i, \beta)$, there cannot be more of such finalists in history (i, b_i) than in $(j, \beta_j(t_j))$; hence $Q_i(b_i, t, \beta) \geq q_i(t, \beta)$. ■

E Details of the No-Tie Theorem

E.1 Proof of Lemma 7

Denote $x := b_*$. Let $\epsilon > 0$. Since the density f_i of $\boldsymbol{t}_i$ is positive on its compact support $[0, \bar{t}_i]$ for every $i \in I$, there exists $\tilde{\eta}(\epsilon) > 0$ such that

$$0 < \eta < \tilde{\eta}(\epsilon) \Longrightarrow \forall i \in I : \forall y \in T_i : \text{Prob}\{y \leq \boldsymbol{t}_i \leq y + 4\eta\} < \frac{\epsilon}{\max_{j \in I} \bar{t}_j}. \tag{73}$$

Pick any $\eta > 0$ such that

$$\eta < \min\left\{\tilde{\eta}(\epsilon), \epsilon, \min_{k\in J}(z_k - a_k)\right\}. \tag{74}$$

For any $\delta > 0$, let

$$N_i^m(x;\delta) := (x-\delta, x+\delta) \cap B_i^m.$$

By Lemma 20 and monotonicity of β_i^*, there exists a $\tilde{\delta}(\epsilon) > 0$ such that

$$\forall i \in I : (\beta_i^*)^{-1}\left(N_i^m(x; 2\tilde{\delta}(\epsilon))\right) \subseteq (a_i - \eta, z_i + \eta). \tag{75}$$

As $\beta^m \to \beta^*$ pointwise almost everywhere, $\beta^m \to \beta^*$ uniformly except on a set $E^* := \Pi_{i\in I} E_i^*$ such that each E_i^* has Lebesgue measure less than η (Littlewood's third principle or Egoroff's theorem). Thus, for any $\delta > 0$ such that

$$\delta \le \tilde{\delta}(\epsilon), \tag{76}$$

there exists $\tilde{m}(\delta)$ with

$$2^{-\tilde{m}(\delta)} < \tilde{\delta}(\epsilon)/2 \tag{77}$$

such that for every integer $m \ge \tilde{m}(\delta)$ and for every $i \in I$, we have

$$\forall t_i \in T_i \setminus E_i^* : |\beta_i^m(t_i) - \beta_i^*(t_i)| < \delta/2, \tag{78}$$

$$(\beta_i^*)^{-1}\left(N_i^m(x; \delta/2)\right) \subseteq (a_i - \eta, z_i + \eta), \tag{79}$$

where (79) follows from (75) and (76).

Now we construct an infinite subsequence $(\beta^{m_n})_{n=1}^\infty$. For each $n = 1, 2, \ldots$, let $\epsilon_n := 1/n$. With ϵ_n taking the role of ϵ, there exists η_n as the left-hand side of Ineq. (74) and $\tilde{\delta}(\epsilon_n)$ specified in Eq. (75). Let

$$\delta_n \;\; := \;\; \min\left\{1/n, \tilde{\delta}(\epsilon_n), x - l\right\}. \tag{80}$$

Hence there exists an $\tilde{m}(\delta_n)$ satisfying Ineq. (77). Let

$$m_n := \min\{m = 1, 2, \ldots : m \ge \tilde{m}(\delta_n); m \ge m_{n-1} + 1\}.$$

Note that $n' > n \Rightarrow m_{n'} > m_n$. Hence subsequence $(\beta^{m_n})_{n=1}^\infty$ is constructed. Also Eqs. (78) and (79) are satisfied when (m_n, δ_n, η_n) plays the role of (m, δ, η).

First, we claim that, for each $i \in I$,

$$\forall t_i \in (\beta_i^*)^{-1}\left(N_i^{m_n}(x;\delta_n/2)\right) \setminus E_i^* : x-\delta_n < \beta_i^{m_n}(t_i) < x+\delta_n, \tag{81}$$

$$(\beta_i^{m_n})^{-1}\left(N_i^{m_n}(x;\delta_n+2^{-m_n})\right) \setminus E_i^* \subseteq (a_i-\eta_n, z_i+\eta_n). \tag{82}$$

To prove (81), pick any $t_i \in (\beta_i^*)^{-1}\left(N_i^{m_n}(x;\delta_n/2)\right) \setminus E_i^*$. Then

$$\beta_i^{m_n}(t_i) \overset{(78)}{<} \beta_i^*(t_i)+\delta_n/2 < x+\delta_n/2+\delta_n/2 = x+\delta_n,$$

and analogously $\beta_i^{m_n}(t_i) > x-\delta_n$. To prove (82), suppose $t_i \le a_i - \eta_n$. Then (75) and monotonicity of β_i^* imply $\beta_i^*(t_i) \le x - 2\tilde{\delta}(\epsilon_n)$; according to (78), either $t_i \in E_i^*$, or

$$\beta_i^{m_n}(t_i) < \beta_i^*(t_i)+\delta_n/2 \le x-2\tilde{\delta}(\epsilon_n)+\delta_n/2 \overset{(76)}{\le} x-2\tilde{\delta}(\epsilon_n)+\tilde{\delta}(\epsilon_n)/2 = x-\tilde{\delta}(\epsilon_n)-\tilde{\delta}(\epsilon_n)/2 < x-\delta_n-2^{-m_n},$$

with the last inequality due to (76) and (77). Analogously, $t_i \ge z_i+\eta_n$ implies either $t_i \in E_i^*$ or $\beta_i^{m_n}(t_i) > x+\delta_n+2^{-m_n}$. Hence (82) follows.

Second, we show that, for each $i \in J$, a_i^n and z_i^n defined by Eqs. (26) and (27) exist and "$a_i < z_i^n$ and $z_i > a_i^n$" holds. By definition of a_i and z_i, $(a_i, z_i) \subseteq (\beta_i^*)^{-1}(N_i^{m_n}(x;\delta_n/2))$. Since the Lebesgue measure of E_i^* is less than η_n, which by (74) is smaller than $z_i - a_i$, there exists $t_i \in (a_i, z_i) \setminus E_i^* \subseteq (\beta_i^*)^{-1}\left(N_i^{m_n}(x;\delta_n/2)\right) \setminus E_i^*$. Hence (81) implies that the sets on the right-hand sides of Eqs. (26) and (27) are nonempty. Thus, a_i^n and z_i^n exist. By the choice of this t_i and Ineq. (78), we have $\beta_i^{m_n}(t_i) < \beta_i^*(t_i)+\delta_n/2 = x+\delta_n/2$. Thus, by definition of z_i^n, $t_i \le z_i^n$. Hence $a_i < z_i^n$, otherwise $t_i > a_i \ge z_i^n$, a contradiction. Analogously, $z_i > a_i^n$.

Third, (28) follows from Eqs. (26) and (27) and the fact that $\beta_i^{m_n}$ is nondecreasing.

Fourth, we prove (30). Recall that $(a_i, z_i) \subseteq (\beta_i^*)^{-1}(N_i^{m_n}(x;\delta_n/2))$ and $a_i^n < z_i$. Thus, if $a_i < a_i^n$ then (81) implies that $(a_i, a_i^n) \subseteq E_i^*$; with E_i^* of Lebesgue measure less than η_n, we have $a_i^n - a_i < \eta_n$. Analogously we have $z_i - z_i^n < \eta_n$. Also, if $a_i^n < a_i - \eta_n$, then (82) implies $(a_i^n, a_i - \eta_n) \subseteq E_i^*$ and hence the interval cannot be longer than η_n; hence $a_i - a_i^n < 2\eta_n$. Analogously we have $z_i^n - z_i < 2\eta_n$. Thus, since $\eta_n < \epsilon_n$ by (74), we have (30).

Fifth, we prove (29) and (31). For any $k \in I \setminus J$, $(\beta_k^*)^{-1}(x)$ is either singleton or empty, hence $a_k = z_k$ by definition. Thus, it follows from (82) that $(\beta_k^{m_n})^{-1}\left(N_k^{m_n}(x;\delta_n+2^{-m_n})\right)$ is either contained in $(a_k-\eta_n, a_k] \cup [z_k, z_k+\eta_n)$ or contained in E_k^*. Since the Lebesgue measure of neither set is bigger than $2\eta_n$, (73) implies (31). Likewise, (73) implies (29) for any $i \in J$ because, by (82), $\{t_i \in T_i : x+\delta_n \le \beta_i^{m_n}(t_i) < x+\delta_n+2^{-m_n}\}$ is contained in $E_i^* \cup [z_i^n, z_i+\eta_n)$.

E.2 Proof of the Decomposition Equation (34)

Eq. (34) is the same as the following equation: for any bids $b_i', b_i \in B_i$ with $b_i' > b_i$,

$$\begin{aligned} & U_i(b_i', t_i, \beta) - U_i(b_i, t_i, \beta) \\ = \;& \mathbb{E}\left[\mathbf{1}\left[b_i' \succ \boldsymbol{t}_{-i}\right]\right]\left(\overline{W}_i(b_i', t_i, \beta) - \overline{W}_i(b_i, t_i, \beta)\right) - (b_i' - b_i)\Pr\left\{b_i' \succ \boldsymbol{t}_{-i}\right\} \\ & + \Pr\left\{b_i' \succ \boldsymbol{t}_{-i}, b_i \not\succ \boldsymbol{t}_{-i}\right\}\left(\overline{W}_i(b_i, t_i, \beta) - b_i - \overline{L}_i(b_i', b_i, t_i, \beta)\right), \end{aligned} \tag{83}$$

where $b_i \succ \boldsymbol{t}_{-i}$ is a shorthand for i's winning event, $b_i \not\succ \boldsymbol{t}_{-i}$ its complement, and

$$\overline{L}_i(b_i', b_i, t_i, \beta) := \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid t_i, \beta) \mid b_i' \succ \boldsymbol{t}_{-i}, b_i \not\succ \boldsymbol{t}_{-i}\right].$$

To prove Eq. (83), note that Eq. (10) says, with the symbols (t_i, β) suppressed,

$$U_i(b_i) = \mathbb{E}\left[\mathbf{1}\left[b_i \succ \boldsymbol{t}_{-i}\right]\left(W_i(\boldsymbol{t}_{-i} \mid b_i) - b_i - L_i(\boldsymbol{t}_{-i})\right)\right] + \mathbb{E}\left[L_i(\boldsymbol{t}_{-i})\right].$$

Then for any bids $b_i' > b_i$,

$$\begin{aligned} U_i(b_i') - U_i(b_i) \;=\;& \mathbb{E}\left[\mathbf{1}\left[b_i' \succ \boldsymbol{t}_{-i}\right] W_i(\boldsymbol{t}_{-i} \mid b_i')\right] - \mathbb{E}\left[\mathbf{1}\left[b_i \succ \boldsymbol{t}_{-i}\right] W_i(\boldsymbol{t}_{-i} \mid b_i)\right] \\ & - b_i'\mathbb{E}\left[\mathbf{1}\left[b_i' \succ \boldsymbol{t}_{-i}\right]\right] + b_i\mathbb{E}\left[\mathbf{1}\left[b_i \succ \boldsymbol{t}_{-i}\right]\right] + \mathbb{E}\left[\left(\mathbf{1}\left[b_i \succ \boldsymbol{t}_{-i}\right] - \mathbf{1}\left[b_i' \succ \boldsymbol{t}_{-i}\right]\right) L_i(\boldsymbol{t}_{-i})\right] \\ \overset{(12)}{=}\;& \mathbb{E}\left[\mathbf{1}\left[b_i' \succ \boldsymbol{t}_{-i}\right]\right]\overline{W}_i(b_i') - \mathbb{E}\left[\mathbf{1}\left[b_i \succ \boldsymbol{t}_{-i}\right]\right]\overline{W}_i(b_i) \\ & - b_i'\mathbb{E}\left[\mathbf{1}\left[b_i' \succ \boldsymbol{t}_{-i}\right]\right] + b_i\mathbb{E}\left[\mathbf{1}\left[b_i \succ \boldsymbol{t}_{-i}\right]\right] - \mathbb{E}\left[\mathbf{1}\left[b_i \not\succ \boldsymbol{t}_{-i}, b_i' \succ \boldsymbol{t}_{-i}\right] L_i(\boldsymbol{t}_{-i})\right]. \end{aligned}$$

Then Eq. (83) follows from breaking apart $\mathbf{1}\left[b_i \succ \boldsymbol{t}_{-i}\right] = \mathbf{1}\left[b_i' \succ \boldsymbol{t}_{-i}\right] - \mathbf{1}\left[b_i \not\succ \boldsymbol{t}_{-i}, b_i' \succ \boldsymbol{t}_{-i}\right]$.

E.3 Proof of Lemma 8

Suppose that the lemma is not true. Then, extracting a subsequence and relabeling superscripts if necessary, we may assume without loss of generality that

$$\lim_{n\to\infty} \Pr\left\{c_j^n < \max_{k\in J\setminus\{j\}} \bar{\beta}_k^n(\boldsymbol{t}_k) \le b_* + \delta_n\right\} > 0. \tag{84}$$

By definition of c_j^n in Eq. (38) and monotonicity of $\bar{\beta}_j^n$,

$$\forall n : \exists \epsilon_n \in \left(0, \min\left\{1/n, \max_{k\in J}(z_k - a_j)/2\right\}\right) : \forall t_j \in (a_j^n, a_j^n + \epsilon_n) : \bar{\beta}_j^n(t_j) = c_j^n. \tag{85}$$

(The above choice of ϵ_n is feasible because by Ineq. (37) $z_k > a_k \ge a_j$ for every $k \in J$.) Let $n \mapsto t_j^n$ be any choice function such that for each n

$$t_j^n \in \left(a_j^n, a_j^n + \epsilon_n\right).$$

We shall derive a contradiction by proving that for some sufficiently large n the type-t_j^n bidder j strictly prefers to deviate from his m_n-equilibrium bid c_j^n to the bid

$$b_j^n := \min\left\{b_j \in B_j^{m_n} : b_j \geq b_* + \delta_n\right\}.$$

To prove this claim, first we establish $\lim_{n\to\infty} \Delta\Pi_j^n(t_j^n) \geq 0$. By Eq. (36), $\Delta\Pi_j^n(t_j^n)$ is equal to a probability times $\overline{W}_j(c_j^n, t_j^n, \bar{\beta}^n) - c_j^n - \overline{L}_j^n(t_j^n)$. Hence it suffices to show $\lim_{n\to\infty}\left(\overline{W}_j(c_j^n, t_j^n, \bar{\beta}^n) - c_j^n - \overline{L}_j^n(t_j^n)\right) \geq 0$. To this end, we first claim that

$$\lim_{n\to\infty} \mathbb{E}\left[L_j(\boldsymbol{t}_{-j} \mid t_j^n, \bar{\beta}^n) \mid \boldsymbol{t}_{-j} \in \Omega_j^n\right] = 0, \tag{86}$$

where $\Omega_j^n := \left\{t_{-j} \in T_{-j} : c_j^n < \max_{k\neq j} \bar{\beta}_k^n(\boldsymbol{t}_k) < b_j^n\right\}$ is the pivotal event of the bid increase. Conditional on Ω_j^n, the winning probability for bidders outside J vanishes as n enlarges (Lemma 7). Thus, Ineq. (86) is unchanged when its integrand $L_j(\boldsymbol{t}_{-j} \mid t_j^n, \bar{\beta}^n)$ is replaced by

$$\sum_{k\in J\setminus\{j\}} \mathbf{1}\left[\bar{\beta}_k^n(\boldsymbol{t}_k) > \max_{j'\notin\{i,j\}} \bar{\beta}_{j'}^n(\boldsymbol{t}_{j'})\right] L_{jk}\left(\boldsymbol{t}_{-j} \mid t_j^n, \bar{\beta}^n\right).$$

For any $k \in J \setminus \{j\}$, bidder j's payoff L_{jk} from losing the auction to bidder k cannot exceed $t_j^n - t_k$. Since $t_j^n < a_j^n + \epsilon_n < a_j^n + 1/n$ by the choice of t_j^n and ϵ_n (Eq. (85)), and since $a_i^n \to_n a_i$ for each $i \in J$, we have for any $t_k \in (a_k^n, z_k^n)$:

$$t_j^n < a_j^n + \epsilon_n \leq a_j + O(1/n) \overset{(37)}{\leq} a_k + O(1/n) \leq a_k^n + O(1/n) < t_k + O(1/n).$$

Thus, $0 \leq L_{jk}^n(t_{-j} \mid t_j^n, \bar{\beta}^n) \leq t_j^n - t_k = O(1/n)$ for any $t_{-j} \in \tilde{\Omega}_{jk}^n$. That proves (86). Thus,

$$\lim_{n\to\infty}\left(\overline{W}_j(c_j^n, t_j^n, \bar{\beta}^n) - c_j^n - \overline{L}_j^n(t_j^n)\right) = \lim_{n\to\infty}\left(\overline{W}_j(c_j^n, t_j^n, \bar{\beta}^n) - c_j^n\right) \geq 0,$$

with the second inequality due to the fact $\bar{\beta}_j^n(t_j^n) = c_j^n$ and Lemma 6. We have therefore proved $\lim_{n\to\infty} \Delta\Pi_j^n(t_j^n) \geq 0$.

Second, by Eq. (85),

$$t_j^n < a_j^n + \epsilon_n \leq a_j + \epsilon_n + O(1/n) < a_j + \max_{k\in J}(z_k - a_j)/2 + O(1/n) = \max_{k\in J}(z_k + a_j)/2 + O(1/n),$$

hence $\lim_{n\to\infty} t_j^n \leq \max_{k\in J}(z_k + a_j)/2 < \max_{k\in J} z_k \leq \max_{k\neq j} z_k$. Thus, Lemma 9 implies $\limsup_{n\to\infty} \Delta W_j^n(t_j^n) > 0$. Plugging this inequality, $\lim_{n\to\infty} \Delta\Pi_j^n(t_j^n) \geq 0$, and $\lim_{n\to\infty}(b_j^n - c_j^n) = 0$ ($c_j^n \in (b_* - \delta_n, b_* + \delta_n)$ since $t_j^n \in (a_j^n, z_j^n)$) into Eq. (34), we have $\limsup_{n\to\infty} \Delta U_j^n(t_j^n) > 0$. Thus, there are sufficiently large n for which the type-t_j^n bidder j strictly prefers deviating to b_j^n from his m_n-equilibrium bid c_j^n. This contradiction proves the lemma.

E.4 A Dominant Rival's Resale Mechanisms

The two lemmas here, where the bid increase is slightly more general than the one specified in the proof of Theorem 2, help us to predict the resale mechanism employed by the dominant bidder j specified in Lemma 8 when he wins with a bid clustered at b_*. In that event, j's winning bid ranges within a neighborhood where his rivals rarely bid, hence his posterior belief about the others stays mostly constant to his winning bid, and so are the posterior virtual utility functions and payment rules at resale. That is formalized by Lemma 21. Lemma 22 further calculates the price markup that j would charge a bidder-type to whom he would resell for sure. We shall use the notation $\mathcal{V}_{k,x}$ defined in Eq. (44).

Lemma 21 *Let b_* be a serious bid and an atom of β_j^*, with $(\bar{\beta}^n)_{n=1}^{\infty}$ specified by Lemma 7. For any $i \neq j$ and any n let $c_i^n \in B_i^{m_n} \cup \{l\}$ and b_i^n specified by Eq. (43) such that $c_i^n < b_i^n$ and Eq. (40) is satisfied. If x^n is someone's winning bid in $[c_i^n, b_i^n]$ for each $n = 1, 2, \ldots$, then for any $k \in I \setminus \{j\}$, with z_k specified in Eq. (25),*

$$\begin{aligned} \lim_{n\to\infty} \left(\bar{\beta}^n\right)^{-1}_{k,\sup}(x^n) &= z_k, & (87)\\ \forall t_k \in T_k : \lim_{n\to\infty} V_k\left(t_k \mid x^n, \bar{\beta}^n\right) &= \mathcal{V}_{k,z_k}(t_k), & (88) \end{aligned}$$

and, if in addition $z_k > \max\left\{t_j, \max_{k'\notin\{j,k\}} \mathcal{V}_{k',z_{k'}}(t_{k'})\right\}$, then

$$\lim_{n\to\infty} p_{k,j,x^n,\bar{\beta}^n}(t_j, t_{-(j,k)}) = \mathcal{V}^{-1}_{k,z_k}\left(\max\left\{t_j, \max_{k'\notin\{j,k\}} \mathcal{V}_{k',z_{k'}}(t_{k'})\right\}\right). \tag{89}$$

Proof Let $k \in I \setminus \{j\}$. Since $x^n \in [c_i^n, b_i^n]$, Eqs. (29), (31) and (40) together imply that the probability measure of the interval between $\left(\bar{\beta}^n\right)^{-1}_{k,\sup}(x^n)$ and z_k^n (defined in Eq. (27)) vanishes. Thus, Eq. (87) follows from the no-gap assumption of F_k and the fact $z_k^n \to_n z_k$ by Eq. (30). To prove (88), note that x^n, a winning bid in the m_n-approximation game, is not an atom of the losers' strategies, due to Eq. (19). Thus, $V_k\left(t_k \mid x^n, \bar{\beta}^n\right)$ obey Eq. (4) if $t_k \le \left(\bar{\beta}^n\right)^{-1}_{k,\sup}(x^n)$, and is equal to $\left(\bar{\beta}^n\right)^{-1}_{k,\sup}(x^n)$ if $t_k \ge \left(\bar{\beta}^n\right)^{-1}_{k,\sup}(x^n)$. Then Eq. (88) follows from Eqs. (44) and (87). To prove Eq. (89), let its condition $z_k > \max\left\{t_j, \max_{k'\notin\{j,k\}} \mathcal{V}_{k',z_{k'}}(t_{k'})\right\}$ be satisfied. Then Eqs. (87) and (88) imply

$$\left(\bar{\beta}^n\right)^{-1}_{i,\sup}(x^n) > \max\left\{t_j, \max_{k\notin\{i,j\}} V_{k,x^n,\bar{\beta}^n}(t_k)\right\}$$

for sufficiently large n. Thus, since j's winning bid is not an atom of $\bar{\beta}^n_{-j}$, the conditions for Eq. (6) are satisfied. Plug Eq. (88) for all $k' \neq j$ into Eq. (6) and we obtain Eq. (89). ■

Lemma 22 *Let b_*, i, j, $(c_i^n, b_i^n)_{n=1}^\infty$ be specified by the hypothesis of Lemma 21 and $(a_k, z_k)_{k\in I}$ by Eqs. (24)–(25). If $t_i^n \to_n t_i$ such that $t_i > a_j$ and $\mathcal{V}_{i,z_i}(t_i) \geq z_k$ for all $k \notin \{i,j\}$ then*

$$\lim_{n\to\infty} \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid t_i^n, \bar\beta^n) \mid \boldsymbol{t}_{-i} \in \Omega_i^n; \boldsymbol{t}_j < t_i^n\right] < t_i - b_*. \tag{90}$$

Proof Let $t_j \in (a_j, \min\{z_j, t_i\})$. Eq. (89) holds for any $k = i$ and $t_{-(i,j)} \in \prod_{k\notin\{i,j\}}[0, z_k)$, as

$$t_i > \max\left\{t_j, \max_{k\notin\{i,j\}} z_k\right\} \geq \max\left\{t_j, \max_{k\notin\{i,j\}} \mathcal{V}_{k,z_k}(t_k)\right\} \tag{91}$$

by hypothesis of the lemma. Integrating Eq. (89) across all such $t_{-(i,j)}$ gives

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{V}_{i,z_i}^{-1}\left(\max\left\{t_j, \max_{k\notin\{i,j\}} \mathcal{V}_{k,z_k}(\boldsymbol{t}_k)\right\}\right)\middle| \boldsymbol{t}_{-(i,j)} \leqq z_{-(i,j)}\right] \\
= \;& \lim_{n\to\infty} \mathbb{E}\left[p_{i,j,\bar\beta_j^n(t_j),\bar\beta^n}\left(t_j, \boldsymbol{t}_{-(i,j)}\right) \mid \boldsymbol{t}_{-(i,j)} \leqq z_{-(i,j)}\right] \\
= \;& \lim_{n\to\infty} \mathbb{E}\left[p_{i,j,\bar\beta_j^n(t_j),\bar\beta^n}\left(t_j, \boldsymbol{t}_{-(i,j)}\right) \middle| \bar\beta_j^n(t_j) > \max_{k\notin\{i,j\}} \bar\beta_k^n(\boldsymbol{t}_k)\right],
\end{aligned}$$

with the second line due to Eq. (87) applied to the case $x^n = \bar\beta_j^n(t_j)$. By Eq. (52),

$$\overline{W}_j\left(\bar\beta_j^n(t_j), t_j, \bar\beta^n\right) = \mathbb{E}\left[\max\left\{t_j, \max_{k\neq j} V_{k,\bar\beta_j^n(t_j),\bar\beta^n}(\boldsymbol{t}_k)\right\}\middle| \bar\beta_j^n(t_j) > \max_{k\neq j} \bar\beta_k^n(\boldsymbol{t}_k)\right].$$

By Eqs. (87) and (88),

$$\lim_{n\to\infty} \overline{W}_j\left(\bar\beta_j^n(t_j), t_j, \bar\beta^n\right) = \mathbb{E}\left[\max\left\{t_j, \max_{k\neq j} \mathcal{V}_{k,z_k}(\boldsymbol{t}_k)\right\}\middle| \boldsymbol{t}_{-j} \leqq z_{-j}\right].$$

Thus,

$$\begin{aligned}
&\lim_{n\to\infty}\left(\mathbb{E}\left[p_{i,j,\bar\beta_j^n(t_j),\bar\beta^n}\left(t_j, \boldsymbol{t}_{-(i,j)}\right)\middle| \bar\beta_j^n(t_j) > \max_{k\notin\{i,j\}} \bar\beta_k^n(\boldsymbol{t}_k)\right] - \overline{W}_j\left(\bar\beta_j^n(t_j), t_j, \bar\beta^n\right)\right) \\
= \;& \mathbb{E}\left[\mathcal{V}_{i,z_i}^{-1}\left(\max\left\{t_j, \max_{k\notin\{i,j\}} \mathcal{V}_{k,z_k}(\boldsymbol{t}_k)\right\}\right)\middle| \boldsymbol{t}_{-(i,j)} \leqq z_{-(i,j)}\right] - \mathbb{E}\left[\max\left\{t_j, \max_{k\neq j} \mathcal{V}_{k,z_k}(\boldsymbol{t}_k)\right\}\middle| \boldsymbol{t}_{-j} \leqq z_{-j}\right],
\end{aligned}$$

which is strictly positive by Ineq. (53) with j playing the role i_* there, since $z_i \geq t_i$ and t_i satisfies Ineq. (91). The above-displayed difference, when its $\overline{W}_j\left(\bar\beta_j^n(t_j), t_j, \bar\beta^n\right)$ is replaced by $\bar\beta_j^n(t_j)$, remains strictly positive because $\bar\beta_j^n(t_j) \leq \overline{W}_j\left(\bar\beta_j^n(t_j), t_j, \bar\beta^n\right)$ by Lemma 6. Thus,

$$\lim_{n\to\infty}\left(\mathbb{E}\left[p_{i,j,\bar\beta_j^n(t_j),\bar\beta^n}\left(t_j, \boldsymbol{t}_{-(i,j)}\right)\middle| \bar\beta_j^n(t_j) > \max_{k\notin\{i,j\}} \bar\beta_k^n(\boldsymbol{t}_k)\right] - \bar\beta_j^n(t_j)\right) > 0.$$

This being true for all $t_j \in (a_j, \min\{z_j, t_i\})$, integrating the inequality across all such t_j and noting that $t_j \in (a_j, z_j)$ implies $\bar\beta_j^n(t_j) \to_n b_*$, we have

$$\lim_{n\to\infty} \mathbb{E}\left[p_{i,j,\bar\beta_j^n(\boldsymbol{t}_j),\bar\beta^n}\left(\boldsymbol{t}_j, \boldsymbol{t}_{-(i,j)}\right)\middle| \bar\beta_j^n(\boldsymbol{t}_j) > \max_{k\notin\{i,j\}} \bar\beta_k^n(\boldsymbol{t}_k); \boldsymbol{t}_j \in [a_j, \min\{z_j, t_i\}]\right] > b_*. \tag{92}$$

With the notation Ω_i^n, defined in Eq. (45), for the pivotal event of the bid increase,

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}\left[L_i(\boldsymbol{t}_{-i}\mid t_i^n,\bar{\beta}^n)\mid \boldsymbol{t}_{-i}\in\Omega_i^n;\boldsymbol{t}_j<t_i^n\right] \\ =\ &\lim_{n\to\infty}\mathbb{E}\left[L_{ij}(\boldsymbol{t}_{-i}\mid t_i^n,\bar{\beta}^n)\,\middle|\,\bar{\beta}_j^n(\boldsymbol{t}_j)>\max_{k\notin\{i,j\}}\bar{\beta}_k^n(\boldsymbol{t}_k);\boldsymbol{t}_j\in[a_j,z_j];\boldsymbol{t}_j\le t_i\right],\end{aligned}\tag{93}$$

where the substitution of L_{ij} for L_i is due to Eqs. (29) and (40), and the substitution in the conditioned event is due to $(a_j^n,z_j^n,t_i^n)\to_n(a_j,z_j,t_i)$ (Eq. (30)). By the hypothesis $\mathcal{V}_{i,z_i}(t_i)\ge\max_{k\notin\{i,j\}}z_k$ and Eq. (88), we have for any $t_j\le(a_j,\min\{z_j,t_i\})$,

$$\lim_{n\to\infty}\left(V_{i,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_i^n)-\max\left\{t_j,\max_{k\notin\{i,j\}}V_{i,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_k)\right\}\right)\ge 0$$

and hence the probability with which the type-t_i^n bidder i wins in j's resale mechanism conditional on the pivotal event goes to one. I.e., the integrand $L_{ij}(\boldsymbol{t}_{-i}\mid t_i^n,\bar{\beta}^n)$ in Eq. (93) can be replaced by $t_i^n-p_{i,j,\bar{\beta}_j^n(\boldsymbol{t}_j),\bar{\beta}^n}\left(\boldsymbol{t}_j,\boldsymbol{t}_{-(i,j)}\right)$ and we have

$$\begin{aligned}&\lim_{n\to\infty}\mathbb{E}\left[L_i(\boldsymbol{t}_{-i}\mid t_i^n,\bar{\beta}^n)\mid \boldsymbol{t}_{-i}\in\Omega_i^n;\boldsymbol{t}_j<t_i^n\right] \\ =\ &\lim_{n\to\infty}\mathbb{E}\left[t_i^n-p_{i,j,\bar{\beta}_j^n(\boldsymbol{t}_j),\bar{\beta}^n}\left(\boldsymbol{t}_j,\boldsymbol{t}_{-(i,j)}\right)\,\middle|\,\bar{\beta}_j^n(\boldsymbol{t}_j)>\max_{k\notin\{i,j\}}\bar{\beta}_k^n(\boldsymbol{t}_k);\boldsymbol{t}_j\in[a_j,\min\{z_j,t_i\}]\right],\end{aligned}$$

which is strictly less than t_i-b_* by Ineq. (92). Hence (90) is proved. ■

E.5 Proof of Lemma 9

Since $b_i^n\ge b_*+\delta_n$ by definition of b_i^n, the probability with which b_i^n wins is no less than $\Pr\left\{b_*+\delta_n>\max_{k\neq i}\bar{\beta}_k^n(\boldsymbol{t}_k)\right\}$, and $\limsup_{n\to\infty}\Pr\left\{b_*+\delta_n>\max_{k\neq i}\bar{\beta}_k^n(\boldsymbol{t}_k)\right\}>0$ by the consequentiality of b_* and the convergence $\bar{\beta}^n\to\beta^*$. Thus, by Eq. (35), it suffices to show

$$\limsup_{n\to\infty}\left(\overline{W}_i(b_i^n,t_i^n,\bar{\beta}^n)-\overline{W}_i(c_i^n,t_i^n,\bar{\beta}^n)\right)>0.$$

To this end, denote $y_k^n:=\left(\bar{\beta}^n\right)_{k,\sup}^{-1}(c_i^n)$ for each $k\in I$. Extracting a convergent subsequence and relabeling superscripts if necessary, we may assume without loss of generality

$$\forall k\in I\ \exists y_k\in T_k:\lim_{n\to\infty}y_k^n=y_k.\tag{94}$$

By Eq. (44), $\mathcal{V}_{k,x}^{-1}(t_i)$ is continuous in x for any $t_i\in[0,x]$. Thus, if $t_i<z_k$ then

$$\forall k\in I\setminus\{i\}:\lim_{n\to\infty}\mathcal{V}_{k,y_k^n}^{-1}(t_i)=\mathcal{V}_{k,y_k}^{-1}(t_i).\tag{95}$$

The rest of the proof uses Proposition 4, with $(y_k^n)_{k\neq i}$ and $(z_k^n)_{k\neq i}$ playing the role of ζ and ζ' there. The proposition is applicable because the winning bids c_i^n and b_i^n are not atoms of $\bar{\beta}_{-i}^n$, hence the posterior virtual utility functions obey Eq. (4). There are only two cases: either (i) $t_i \geq \max_{j\neq i} y_j$ or (ii) $t_i < y_j$ for some $j \neq i$.

Case (i): This implies, by Eq. (94), $t_i^n + 1/n \geq \max_{j\neq i} y_j^n$ for sufficiently large n. Since $t_i < \max_{k\neq i} z_k$ by hypothesis of the lemma, Eq. (30) implies $t_i^n < \max_{k\neq i} z_k^n$ for sufficiently large n. Thus, $\max_{j\neq i} y_j^n - 1/n \leq t_i^n < \max_{k\neq i} z_k^n$ for sufficiently large n. For any such n, $\overline{W}_i(c_i^n, t_i^n, \bar{\beta}^n) \leq t_i^n + 1/n$ because the public history (i, c_i^n) implies $\boldsymbol{t}_j \leq y_j^n$ for each $j \neq i$. Thus, mimicking the reasoning for Ineq. (55), we have, for some $j \neq i$ with $z_j > t_i$,

$$\overline{W}_i(b_i^n, t_i^n, \bar{\beta}^n) - \overline{W}_i(c_i^n, t_i^n, \bar{\beta}^n) \geq \left(1 - \frac{F_j\left(\mathcal{V}_{j,z_j^n}^{-1}(t_i^n)\right)}{F_j(z_j^n)}\right)\left(\mathcal{V}_{j,z_j^n}^{-1}(t_i^n) - t_i^n - 1/n\right).$$

Since $t_i^n \to t_i$ and $z_i^n \to_n z_i$, Eq. (95) and the continuity of $\mathcal{V}_{j,z_j^n}^{-1}$ together imply that the right-hand side converges to $\left(1 - F_j\left(\mathcal{V}_{j,z_j}^{-1}(t_i)\right)/F_j(z_j)\right)\left(\mathcal{V}_{j,z_j}^{-1}(t_i) - t_i\right)$, which is strictly positive since $\mathcal{V}_{j,z_j}^{-1}(t_i) > t_i$ due to $z_j > t_i$ (which implies $z_j > \mathcal{V}_{j,y_j}^{-1} > t_i$ by Eq. (44)).

Case (ii): By Eq. (94), for infinitely many n, $t_i^n < y_j^n$ and hence Ineq. (56) holds, i.e.,

$$\overline{W}_i(b_i^n, t_i^n, \bar{\beta}^n) - \overline{W}_i(c_i^n, t_i^n, \bar{\beta}^n) \geq \gamma_n \Pr\left\{c_i^n < \max_{j\in J\setminus\{i\}} \bar{\beta}_j^n(\boldsymbol{t}_j) < b_i^n\right\},$$

where according to Eq. (57)

$$\gamma_n = \frac{\prod_{k\neq i} F_k(y_k^n) - \prod_{k\neq i} F_k\left(\min\left\{\mathcal{V}_{k,y_k^n}^{-1}(t_i^n), y_k^n\right\}\right)}{\left(\prod_{k\neq i} F_k(z_k^n)\right)\left(\prod_{k\neq i} F_k(y_k^n)\right)} t_i^n.$$

By Ineq. (41), $\limsup_{n\to\infty} \Pr\left\{c_i^n < \max_{j\in J\setminus\{i\}} \bar{\beta}_j^n(\boldsymbol{t}_j) < b_i^n\right\} > 0$; by Eq. (95),

$$\limsup_{n\to\infty} \gamma_n = \frac{\prod_{k\neq i} F_k(y_k) - \prod_{k\neq i} F_k\left(\min\left\{\mathcal{V}_{k,y_k}^{-1}(t_i), y_k\right\}\right)}{\left(\prod_{k\neq i} F_k(z_k)\right)\left(\prod_{k\neq i} F_k(y_k)\right)} t_i > 0,$$

where the inequality is due to the fact $t_i < y_j$ (which implies $t_i < \mathcal{V}_{j,y_j}^{-1} < y_j$ by Eq. (44)). Hence again $\limsup_{n\to\infty}\left(\overline{W}_i(b_i^n, t_i^n, \bar{\beta}^n) - \overline{W}_i(c_i^n, t_i^n, \bar{\beta}^n)\right) > 0$, as desired.

E.6 Proof of Lemma 10

By Eqs. (36), (45) and (46), the pivotal effect equals $\Delta\Pi_i^n(t_i^n) = \Pr(\Omega_i^n)\psi_i^n(t_i^n)$. By Ineq. (41), $\lim_n \Pr(\Omega_i^n) > 0$. Thus, it suffices to show $\lim_n \psi_i^n(t_i^n) > 0$. To this end, note that Lemma 22 is applicable. Furthermore, since $(z_j^n, t_i^n) \to_n (z_j, t_i)$ and $z_j \le t_i$ by hypothesis of this lemma, $\lim_n \Pr\{\boldsymbol{t}_j < t_i^n \mid \boldsymbol{t}_{-i} \in \Omega_i^n\} = 1$. Hence

$$\lim_{n\to\infty} \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid t_i^n, \bar{\beta}^n) \mid \boldsymbol{t}_{-i} \in \Omega_i^n\right] = \lim_{n\to\infty} \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid t_i^n, \bar{\beta}^n) \mid \boldsymbol{t}_{-i} \in \Omega_i^n; \boldsymbol{t}_j < t_i^n\right] < t_i - b_*$$

by Ineq. (90). Thus, by the fact that $\overline{W}_i\left(c_i^n, t_i^n, \bar{\beta}^n\right) \ge t_i^n$ and $c_i^n \to_n b_*$ (Ineq. (42)),

$$\lim_{n\to\infty} \left(\overline{W}_i\left(c_i^n, t_i^n, \bar{\beta}^n\right) - c_i^n - \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid t_i^n, \bar{\beta}^n) \mid \boldsymbol{t}_{-i} \in \Omega_i^n\right]\right) > t_i - b_* - (t_i - b_*) = 0,$$

which means, by Eq. (46), that $\lim_n \psi_i^n(t_i^n) > 0$, as desired.

E.7 Proof of Lemma 11

Step 1: A resale mechanism for bidder i In the m_n-approximation game, bidder i upon winning can offer resale via the following game form M^n for bidders $k \ne i$, with the notation $(z_i^n, \mathcal{V}_{i,z_i^n})$ defined in Eqs. (27) and (44):

a. Every bidder $k \ne i$ picks an element from $[0, z_k^n]$, say t_k, and reports it as k's type.

b. If $t_j \ge \mathcal{V}_{i,z_i^n}(t_i^n)$ then—

   i. i resells the good to a bidder $k \in I \setminus \{i\}$ for whom

   $$V_{k,c_i^n,\bar{\beta}^n}(t_k) \ge \max\left\{t_i^n, \max_{h\notin\{i,k\}} V_{h,,c_i^n,\bar{\beta}^n}(t_h)\right\}$$

   at the price $V^{-1}_{k,c_i^n,\bar{\beta}^n}\left(\max\left\{t_i^n, \max_{h\notin\{i,k\}} V_{h,,c_i^n,\bar{\beta}^n}(t_h)\right\}\right)$;

   ii. if no such k exists then i keeps the good.

c. If $t_j < \mathcal{V}_{i,z_i^n}(t_i^n)$ then—

   i. i resells the good to a bidder $k \in I \setminus \{i, j\}$ for whom

   $$V_{k,c_i^n,\bar{\beta}^n}(t_k) \ge \max\left\{\mathcal{V}_{i,z_i^n}(t_i^n), \max_{h\notin\{i,j,k\}} V_{h,,c_i^n,\bar{\beta}^n}(t_h)\right\}$$

   at the price $V^{-1}_{k,c_i^n,\bar{\beta}^n}\left(\max\left\{\mathcal{V}_{i,z_i^n}(t_i^n), \max_{h\notin\{i,j,k\}} V_{h,,c_i^n,\bar{\beta}^n}(t_h)\right\}\right)$;

   ii. if no such k exists then i keeps the good.

We claim that the mechanism M^n is ex post incentive feasible. It is ex post individually rational because a bidder $k \neq i$ can stay out by reporting his type being zero, thereby reporting a negative virtual utility. This gives him zero probability to win in Case (b), as $t_i^n \geq 0$. This also renders zero winning probability in Case (c), where $t_j \leq \mathcal{V}_{i,z_i^n}(t_i^n)$ implies $\mathcal{V}_{i,z_i^n}(t_i^n) \geq 0$. Thus, in either case bidder k can stay out thereby ensuring zero payoff.

The mechanism M^n is also ex post incentive compatible. For any $k \neq i$ and any t_{-k}, if bidder k's true type is t_k, then by the rules in (b) and (c) his payoff conditional on winning is positive if and only if $t_k > V^{-1}_{k,c_i^n,\bar{\beta}^n}(v_k^n(t_{-k}))$, where

$$v_k^n(t_{-k}) := \begin{cases} \max\left\{t_i^n, \max_{h \notin \{i,k\}} V_{h,,c_i^n,\bar{\beta}^n}(t_h)\right\} & \text{if } t_j \geq \mathcal{V}_{i,z_i^n}(t_i^n) \\ \max\left\{\mathcal{V}_{i,z_i^n}(t_i^n), \max_{h \notin \{i,j,k\}} V_{h,,c_i^n,\bar{\beta}^n}(t_h)\right\} & \text{if } t_j < \mathcal{V}_{i,z_i^n}(t_i^n). \end{cases}$$

With $V_{k,c_i^n,\bar{\beta}^n}$ strictly increasing, $t_k > V^{-1}_{k,c_i^n,\bar{\beta}^n}(v_k^n(t_{-k}))$ is equivalent to $V_{k,c_i^n,\bar{\beta}^n}(t_k) > v_k^n(t_{-k})$, i.e., the event that k wins in M^n after reporting truthfully. Thus, having a positive payoff from winning in M^n is equivalent to the event that he should win after truthtelling. Since the payoff from not winning in M^n is equal to zero, this implies incentive compatibility for any bidder $k \in I \setminus \{i\}$.

Step 2: Bidder i's expected payoff as a reseller Denote $\hat{w}^n(t_{-i})$ for i's ex post payoff generated by the participation and truthtelling equilibrium in M^n when the realized type profile across $k \neq i$ is t_{-i}. By revealed preference,

$$\overline{W}_i\left(c_i^n, t_i^n, \bar{\beta}^n\right) \geq \mathbb{E}\left[\hat{w}^n(\boldsymbol{t}_{-i}) \,\middle|\, \max_{k \neq i} \bar{\beta}_k^n(\boldsymbol{t}_k) < c_i^n\right].$$

Denote

$$\begin{aligned} X^n &:= \mathbb{E}\left[\mathbf{1}\left[\boldsymbol{t}_j \geq \mathcal{V}_{i,z_i^n}(t_i^n)\right]\left(\hat{w}^n\left(\boldsymbol{t}_{-i}\right) - c_i^n\right) \,\middle|\, \max_{k \neq i} \bar{\beta}_k^n(\boldsymbol{t}_k) < c_i^n\right], \\ Y^n &:= \mathbb{E}\left[\mathbf{1}\left[\boldsymbol{t}_j < \mathcal{V}_{i,z_i^n}(t_i^n)\right]\left(\hat{w}^n\left(\boldsymbol{t}_{-i}\right) - c_i^n\right) \,\middle|\, \max_{k \neq i} \bar{\beta}_k^n(\boldsymbol{t}_k) < c_i^n\right]. \end{aligned}$$

Then

$$\overline{W}_i\left(c_i^n, t_i^n, \bar{\beta}^n\right) - c_i^n \geq X^n + Y^n.$$

To calculate X^n, note from its definition that it is an integral on the set of t_{-i} such that $t_j \geq \mathcal{V}_{i,z_i^n}(t_i^n)$. At such t_{-i}, mechanism M^n follows its rule (b), which coincides with the resale mechanism $M_i(c_i^n, t_i^n, \bar{\beta}^n)$ that the type-t_i^n bidder i would choose upon winning with bid c_i^n. (The posterior virtual utility functions conditional on the public history (i, c_i^n) are $\left(V_{k,c_i^n,\bar\beta}\right)_{k\neq i}$ because Lemma 2.a.i applies, as c_i^n is not an atom of $\bar\beta_{-i}^n$.) Thus,

$$X^n = \mathbb{E}\left[\mathbf{1}\left[\boldsymbol{t}_j \geq \mathcal{V}_{i,z_i^n}(t_i^n)\right]\left(W_i\left(\boldsymbol{t}_{-i} \mid c_i^n, t_i^n, \bar\beta^n\right) - c_i^n\right)\Big|\max_{k\neq i}\bar\beta_k^n(\boldsymbol{t}_k) < c_i^n\right].$$

If $\Pr\left\{\boldsymbol{t}_j \geq \mathcal{V}_{i,z_i^n}(t_i^n)\,\middle|\max_{k\neq i}\bar\beta_k^n(\boldsymbol{t}_k) < c_i^n\right\} = 0$ then $X^n = 0$; else then Lemma 19 implies

$$\mathbb{E}\left[W_i\left(\boldsymbol{t}_{-i} \mid c_i^n, t_i^n, \bar\beta^n\right)\Big|\boldsymbol{t}_j \geq \mathcal{V}_{i,z_i^n}(t_i^n); \max_{k\neq i}\bar\beta_k^n(\boldsymbol{t}_k) < c_i^n\right] \geq \mathbb{E}\left[W_i\left(\boldsymbol{t}_{-i} \mid c_i^n, t_i^n, \bar\beta^n\right)\Big|\max_{k\neq i}\bar\beta_k^n(\boldsymbol{t}_k) < c_i^n\right].$$

Thus,

$$X^n \;\;\geq\;\; \Pr\left\{\boldsymbol{t}_j \geq \mathcal{V}_{i,z_i^n}(t_i^n)\,\Big|\max_{k\neq i}\bar\beta_k^n(\boldsymbol{t}_k) < c_i^n\right\}\mathbb{E}\left[W_i\left(\boldsymbol{t}_{-i} \mid c_i^n, t_i^n, \bar\beta^n\right) - c_i^n\,\Big|\max_{k\neq i}\bar\beta_k^n(\boldsymbol{t}_k) < c_i^n\right];$$

furthermore, since $\bar\beta_i^n(t_i^n) = c_i^n$, the second factor on the right-hand side according to Lemma 6 is nonnegative. Thus, $X^n \geq 0$ and hence

$$\overline{W}_i\left(c_i^n, t_i^n, \bar\beta^n\right) - c_i^n \geq Y^n. \tag{96}$$

To calculate Y^n, note from its definition that it is an integral on the set of t_{-i} such that $t_j < \mathcal{V}_{i,z_i^n}(t_i^n)$. At such t_{-i}, mechanism M^n follows its rule (c), under which t_j has no effect on the outcome of M^n, hence i's ex post payoff $\hat w^n(t_{-i})$ from M^n is constant to t_j. Since the indicator $\mathbf{1}\left[\boldsymbol{t}_j < \mathcal{V}_{i,z_i^n}(t_i^n)\right]$ is weakly decreasing in t_j, with $\hat w^n(t_{-i})$ independent of t_j and nonnegative, $\mathbf{1}\left[t_j < \mathcal{V}_{i,z_i^n}(t_i^n)\right]\hat w^n(t_{-i})$ is a weakly decreasing function of t_j for any $t_{-(i,j)}$. Hence the integral of this function cannot increase when we move upward some mass of $\boldsymbol{t}_j$, by replacing the conditioned event $\max_{k\neq i}\bar\beta_k^n(\boldsymbol{t}_k) < c_i^n$ with the one in the following:

$$Y^n \geq \mathbb{E}\left[\mathbf{1}\left[\boldsymbol{t}_j < \mathcal{V}_{i,z_i^n}(t_i^n)\right]\left(\hat w^n\left(\boldsymbol{t}_{-i}\right) - c_i^n\right)\Big|c_i^n < \bar\beta_j^n(\boldsymbol{t}_j) < b_i^n; \max_{k\notin\{i,j\}}\bar\beta_k^n(\boldsymbol{t}_k) < c_i^n\right].$$

By Eq. (40), we can replace the above conditioned event by Ω_i^n defined in Eq. (45). Thus,

$$\lim_{n\to\infty} Y^n \geq \lim_{n\to\infty}\mathbb{E}\left[\mathbf{1}\left[\boldsymbol{t}_j < \mathcal{V}_{i,z_i^n}(t_i^n)\right]\left(\hat w^n\left(\boldsymbol{t}_{-i}\right) - c_i^n\right) \mid \boldsymbol{t}_{-i} \in \Omega_i^n\right]. \tag{97}$$

Step 3: The winner's curse By Eq. (40), the probability with which bidders $k \notin \{i, j\}$ wins given Ω_i^n vanishes as $n \to \infty$. Thus, the loser's payoff for i comes mainly from i's payoff from losing to j, i.e., when bidder j with some type t_j wins with bid $\bar\beta_j^n(t_j) \in (c_i^n, b_i^n)$. In that event, bidder j chooses the resale mechanism $M_j\left(\bar\beta_j^n(t_j), t_j, \bar\beta^n\right)$, which is determined by posterior virtual utility functions $\left(V_{k,\bar\beta_j^n(t_j),\bar\beta^n}\right)_{k\neq j}$ (due to Lemma 2.a.i, applicable because the winning bid $\bar\beta_j^n(t_j)$ is not an atom of $\bar\beta_{-j}^n$, by Eq. (19)). Recall that the probability with which bidder i gets to buy the good from j is denoted by $q_{ij}\left(t_i^n, t_{-i}, \bar\beta^n\right)$, with the price denoted by $p_{i,j,\bar\beta_j^n(t_j),\bar\beta^n}(t_{-i})$. Thus,

$$
\begin{aligned}
& \lim_{n\to\infty} \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid t_i^n, \bar\beta^n) \mid \boldsymbol{t}_{-i} \in \Omega_i^n\right] \\
\overset{(40)}{=} \;& \lim_{n\to\infty} \mathbb{E}\left[\left(t_i^n - p_{i,j,\bar\beta_j^n(t_j),\bar\beta^n}(\boldsymbol{t}_{-i})\right) q_{ij}\left(t_i^n, \boldsymbol{t}_{-i}, \bar\beta^n\right)\middle| \boldsymbol{t}_{-i} \in \Omega_i^n\right] \qquad (98)\\
= \;& \lim_{n\to\infty} \mathbb{E}\left[\mathbf{1}\left[\boldsymbol{t}_j < \mathcal{V}_{i,z_i^n}(t_i^n)\right]\left(t_i^n - p_{i,j,\bar\beta_j^n(t_j),\bar\beta^n}(\boldsymbol{t}_{-i})\right) q_{ij}\left(t_i^n, \boldsymbol{t}_{-i}, \bar\beta^n\right)\middle| \boldsymbol{t}_{-i} \in \Omega_i^n\right],
\end{aligned}
$$

where the second equality holds because $\boldsymbol{t}_j \geq \mathcal{V}_{i,z_i^n}(t_i^n)$ implies that the probability with which bidder i can buy the good from j, and hence i's payoff at resale, vanishes as n enlarges.

Step 4: Y^n balances the winner's curse Combining (97) with (98) yields

$$
\begin{aligned}
& \lim_{n\to\infty}\left(Y^n - \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid t_i^n, \bar\beta^n) \mid \boldsymbol{t}_{-i} \in \Omega_i^n\right]\right) \\
\geq \;& \lim_{n\to\infty} \mathbb{E}\left[\mathbf{1}\left[\boldsymbol{t}_j < \mathcal{V}_{i,z_i^n}(t_i^n)\right]\left(\hat w^n\left(t_i^n, \boldsymbol{t}_{-i}\right) - c_i^n - \left(t_i^n - p_{i,j,\bar\beta_j^n(t_j),\bar\beta^n}(\boldsymbol{t}_{-i})\right) q_{ij}\left(t_i^n, \boldsymbol{t}_{-i}, \bar\beta^n\right)\right)\middle| \boldsymbol{t}_{-i} \in \Omega_i^n\right].
\end{aligned}
$$

To calculate the right-hand side, let t_{-i} range within Ω_i^n such that $t_j < \mathcal{V}_{i,z_i^n}(t_i^n)$. Then mechanism M^n operates under rule (c), and i's payoff $\hat w^n\left(t_i^n, \boldsymbol{t}_{-i}\right)$ is equal to either t_i^n if

$$
\mathcal{V}_{i,z_i^n}(t_i^n) > \max_{k\notin\{i,j\}} V_{k,c_i^n,\bar\beta^n}(t_k), \tag{99}
$$

or the payment

$$
V_{k,c_i^n,\bar\beta^n}^{-1}\left(\max\left\{\mathcal{V}_{i,z_i^n}(t_i^n), \max_{h\notin\{i,j,k\}} V_{h,,c_i^n,\bar\beta^n}(t_h)\right\}\right) \tag{100}
$$

from some bidder $k \notin \{i,j\}$ if

$$
V_{k,c_i^n,\bar\beta^n}(t_k) > \max\left\{\mathcal{V}_{i,z_i^n}(t_i^n), \max_{h\notin\{i,j,k\}} V_{h,,c_i^n,\bar\beta^n}(t_h)\right\}. \tag{101}
$$

By Eq. (88) and $z_i^n \to_n z_i$, $t_i^n \to_n t_i$ and continuity of the mapping $x \mapsto \mathcal{V}_{i,x}(t_i)$, we have

$$
\begin{gathered}
\lim_{n\to\infty} \mathcal{V}_{i,z_i^n}(t_i) = \mathcal{V}_{i,z_i}(t_i), \\
\lim_{n\to\infty} V_{k',\bar\beta_j^n(t_j),\bar\beta^n}(t_{k'}) = \mathcal{V}_{k',z_{k'}}(t_{k'}) = \lim_{n\to\infty} V_{k',c_i^n,\bar\beta^n}(t_{k'}), \\
\lim_{n\to\infty} V_{i,\bar\beta_j^n(t_j),\bar\beta^n}(t_i^n) = \mathcal{V}_{i,z_i}(t_i) = \lim_{n\to\infty} \mathcal{V}_{i,z_i^n}(t_i^n).
\end{gathered}
$$

Then for all sufficiently large n, the event (99) is approximated by

$$
V_{i,\bar\beta_j^n(t_j),\bar\beta^n}(t_i^n) > \max_{k\notin\{i,j\}} V_{k,\bar\beta_j^n(t_j),\bar\beta^n}(t_k)
$$

(which means if j wins j would resell to i since $t_j < \mathcal{V}_{i,z_i^n}(t_i^n) \approx V_{i,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_i^n)$), and the event (101) is approximated by

$$V_{k,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_k) > \max\left\{V_{i,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_i^n), \max_{h\notin\{i,j,k\}} V_{h,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_h)\right\}$$

(which means j would resell to k), with the payment (100) approximated by $p_{k,j,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_{-k})$. Thus, for all $t_{-i} \in \Omega_i^n$ with $t_j < \mathcal{V}_{i,z_i^n}(t_i^n)$ except a subset whose measure is $O(1/n)$,

$$\hat{w}^n\left(t_i^n, t_{-i}\right) + O(1/n) = t_i^n q_{ij}(t_i^n, t_{-k}, \bar{\beta}^n) + \sum_{k\in I\setminus\{i,j\}} q_{kj}(t_k, t_{-k}, \bar{\beta}^n) p_{k,j,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_{-k})$$

and hence

$$\begin{aligned}
&\hat{w}^n\left(t_i^n, t_{-i}\right) - \left(t_i^n - p_{i,j,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_{-i})\right) q_{ij}\left(t_i^n, t_{-i}, \bar{\beta}^n\right) + O(1/n) \\
= &\sum_{k\in I\setminus\{j\}} q_{kj}(t_k, t_{-k}, \bar{\beta}^n) p_{k,j,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_{-k}),
\end{aligned}$$

which is equal to $W_j\left(t_i^n, t_{-(i,j)} \mid \bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n\right)$ because $t_j < \mathcal{V}_{i,z_i^n}(t_i^n)$ implies $t_j < V_{i,\bar{\beta}_j^n(t_j),\bar{\beta}^n}(t_i^n)$ for all sufficiently large n, at which bidder j, upon winning, always resells the good. Thus,

$$\begin{aligned}
&\lim_{n\to\infty}\left(Y^n - \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid t_i^n, \bar{\beta}^n) \mid \boldsymbol{t}_{-i} \in \Omega_i^n\right]\right) \\
\geq &\lim_{n\to\infty} \mathbb{E}\left[\mathbf{1}\left[\boldsymbol{t}_j < \mathcal{V}_{i,z_i^n}(t_i^n)\right]\left(W_j\left(t_i^n, \boldsymbol{t}_{-(i,j)} \mid \bar{\beta}_j^n(\boldsymbol{t}_j), \boldsymbol{t}_j, \bar{\beta}^n\right) - c_i^n\right)\Big| \boldsymbol{t}_{-i} \in \Omega_i^n\right].
\end{aligned}$$

Combining this with Eq. (96), as well as the facts $\lim_{n\to\infty} c_i^n = b_*$ by Eq. (42) and $b_* = \lim_{n\to\infty} \bar{\beta}_j^n(t_j)$ for all t_j such that $\bar{\beta}_j^n(t_j) \in (c_i^n, b_i^n)$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\left(\overline{W}_i\left(c_i^n, t_i^n, \bar{\beta}^n\right) - c_i^n - \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid t_i^n, \bar{\beta}^n) \mid \boldsymbol{t}_{-i} \in \Omega_i^n\right]\right) \\
\geq &\lim_{n\to\infty} \mathbb{E}\left[\mathbf{1}\left[\boldsymbol{t}_j < \mathcal{V}_{i,z_i^n}(t_i^n)\right]\left(W_j\left(t_i^n, \boldsymbol{t}_{-(i,j)} \mid \bar{\beta}_j^n(\boldsymbol{t}_j), \boldsymbol{t}_j, \bar{\beta}^n\right) - \bar{\beta}_j^n(\boldsymbol{t}_j)\right)\Big| \boldsymbol{t}_{-i} \in \Omega_i^n\right],
\end{aligned}$$

which is Eq. (47). This proves the lemma.

E.8 Proof of Lemma 12

Pick any $t_j \in (a_j, z_j)$. By Lemma 6,

$$\bar{\beta}_j^n(t_j) \quad \leq \quad \mathbb{E}\left[W_j\left(\boldsymbol{t}_{-j} \mid \bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n\right)\Big| \max_{k\neq j} \bar{\beta}_k^n(\boldsymbol{t}_k) < \bar{\beta}_j^n(t_j)\right].$$

Taking the limit and using Eq. (87), we have

$$\lim_{n\to\infty} \bar{\beta}_j^n(t_j) \quad \leq \quad \mathbb{E}_{\boldsymbol{t}_i}\left[\lim_{n\to\infty} \mathbb{E}_{\boldsymbol{t}_{-(i,j)}}\left[W_j\left(\boldsymbol{t}_{-j} \mid \bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n\right)\Big| \max_{k\notin\{i,j\}} \bar{\beta}_k^n(\boldsymbol{t}_k) < \bar{\beta}_j^n(t_j)\right]\Big| \boldsymbol{t}_i \leq z_i\right].$$

By Eq. (19) the winning bid $\bar{\beta}_j^n(t_j)$ is not an atom of $\bar{\beta}_{-j}^n$, so the $W_j\left(t_{-j} \mid \bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n\right)$ on the right-hand side obeys Eq. (7) and hence is determined by the virtual utility functions $\left(V_{k,\bar{\beta}_j^n(t_j),\bar{\beta}^n}\right)_{k\neq j}$. By Lemma 21, for each $k \neq j$, $V_{k,\bar{\beta}_j^n(t_j),\bar{\beta}^n} \to_n \mathcal{V}_{k,z_k}$, which is the virtual utility function given distribution $F_k(\cdot)/F_k(z_k)$. Thus, $W_j\left(t_{-j} \mid \bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n\right)$ converges to the type-t_j reseller j's expected payoff from the Myerson auction $\mathbb{M}(z, t_j)$ defined in §A.1. I.e., with the φ_i defined in Eq. (61) where ζ_k and α_i are z_k here,

$$\lim_{n\to\infty} \mathbb{E}_{\boldsymbol{t}_{-(i,j)}}\left[W_j\left(t_i, \boldsymbol{t}_{-(i,j)} \mid \bar{\beta}_j^n(t_j), t_j, \bar{\beta}^n\right) \middle| \max_{k\notin\{i,j\}} \bar{\beta}_k^n(\boldsymbol{t}_k) < \bar{\beta}_j^n(t_j)\right] = \varphi_i(t_i, t_j). \tag{102}$$

Denote $\overline{\varphi}_i(t_j) := \mathbb{E}_{\boldsymbol{t}_i}\left[\varphi_i(\boldsymbol{t}_i, t_j) \mid \boldsymbol{t}_i \in [0, z_i]\right]$, so the above-displayed inequality means

$$\lim_{n\to\infty} \bar{\beta}_j^n(t_j) \leq \overline{\varphi}_i(t_j). \tag{103}$$

Denote

$$\begin{aligned} \varphi_*(t_i) &:= \mathbb{E}\left[\mathbf{1}[\boldsymbol{t}_j < z_i]\varphi_i(t_i, \boldsymbol{t}_j) \mid \boldsymbol{t}_j \in (a_j, z_j)\right], \\ \overline{\varphi}_* &:= \mathbb{E}\left[\mathbf{1}[\boldsymbol{t}_j < z_i]\overline{\varphi}_i(\boldsymbol{t}_j) \mid \boldsymbol{t}_j \in (a_j, z_j)\right]. \end{aligned}$$

Note that $\varphi_*(z_i) \geq \overline{\varphi}_*$. Otherwise, by continuity of φ_* (Corollary 3), $\varphi_* < \overline{\varphi}_*$ on an interval $(\theta, z_i]$ for some $\theta < z_i$, which contradicts Ineq. (63) of Corollary 3, with $(i, j, (a_j, \min\{z_i, z_i\}))$ being (i, i_*, S) there. Plugging the definitions of φ_* and $\overline{\varphi}_*$ into $\varphi_*(z_i) \geq \overline{\varphi}_*$, we have

$$\mathbb{E}_{\boldsymbol{t}_j}\left[\mathbf{1}\left[\boldsymbol{t}_j < z_i\right]\left(\varphi_i(z_i, \boldsymbol{t}_j) - \overline{\varphi}_i(\boldsymbol{t}_j)\right) \middle| \boldsymbol{t}_j \in (a_j, z_j)\right] \geq 0.$$

Replace $\varphi_i(z_i, t_j)$ with the left-hand side of Eq. (102) for the case $t_i = z_i$, switch the positions of the integration and the limit operators and then use Ineq. (103) to obtain

$$\lim_{n\to\infty} \mathbb{E}\left[\mathbf{1}\left[\boldsymbol{t}_j < z_i\right]\left(W_j\left(z_i, \boldsymbol{t}_{-(i,j)} \mid \bar{\beta}_j^n(\boldsymbol{t}_j), \boldsymbol{t}_j, \bar{\beta}^n\right) - \bar{\beta}_j^n(\boldsymbol{t}_j)\right) \middle| \begin{array}{l} \boldsymbol{t}_j \in (a_j, z_j); \\ \max_{k\notin\{i,j\}} \bar{\beta}_k^n(\boldsymbol{t}_k) < \bar{\beta}_j^n(\boldsymbol{t}_j) \end{array}\right] \geq 0. \tag{104}$$

We construct a sequence $(t_i^n)_{n=1}^\infty$ converging to z_i such that $\bar{\beta}_i^n(t_i^n) = c_i^n$ for each n and Ineq. (48) holds. To this end, first recall $\left(\bar{\beta}^n\right)_i^{-1}(c_i^n) \neq \varnothing$ by Eq. (39). By Eq. (87), $\left(\bar{\beta}^n\right)_{i,\sup}^{-1}(c_i^n)$ the supremum of this inverse image converges to z_i as $n \to \infty$. Thus, there exists $(t_i^n)_{n=1}^\infty$ such that $t_i^n \in \left(\bar{\beta}^n\right)_i^{-1}(c_i^n)$ for each n and $t_i^n \to_n z_i$. Second, denote

$$h^n(t_i) := \mathbb{E}\left[\mathbf{1}\left[\boldsymbol{t}_j < z_i\right] W_j\left(t_i, \boldsymbol{t}_{-(i,j)} \mid \bar{\beta}_j^n(\boldsymbol{t}_j), \boldsymbol{t}_j, \bar{\beta}^n\right) \middle| \begin{array}{l} \boldsymbol{t}_j \in (a_j, z_j); \\ \max_{k\notin\{i,j\}} \bar{\beta}_k^n(\boldsymbol{t}_k) < \bar{\beta}_j^n(\boldsymbol{t}_j) \end{array}\right].$$

Note that $(h^n)_{n=1}^\infty$ is equicontinuous at z_i by Corollary 2, with (i, j, z^n) being (i, i_*, ζ^n) there. The corollary is applicable because the integration domain is a subset in T_{-i} of strictly positive measure and the integrand $W_j\left(t_i, t_{-(i,j)} \mid \bar\beta_j^n(t_j), t_j, \bar\beta^n\right)$ is reseller j's ex post payoff in the regular (hence semi-regular) resale environment ζ^n. Thus, there exists a subsequence $(t_i^{n_\gamma})_{\gamma=1}^\infty$ such that $h^n(t_i^{n_\gamma}) \geq h^n(z_i) - 1/\gamma$ for any $\gamma \in \{1, 2, \ldots\}$. Plugging this into (104), we have

$$\lim_{\gamma\to\infty} \mathbb{E}\left[\mathbf{1}\left[\boldsymbol{t}_j < z_i\right]\left(W_j\left(t_i^{n_\gamma}, \boldsymbol{t}_{-(i,j)} \mid \bar\beta_j^{n_\gamma}(\boldsymbol{t}_j), \boldsymbol{t}_j, \bar\beta^{n_\gamma}\right) - \bar\beta_j^{n_\gamma}(\boldsymbol{t}_j)\right) \,\middle|\, \begin{array}{l} \boldsymbol{t}_j \in (a_j, z_j); \\ \max_{k\notin\{i,j\}} \bar\beta_k^{n_\gamma}(\boldsymbol{t}_k) < \bar\beta_j^{n_\gamma}(\boldsymbol{t}_j) \end{array}\right] \geq 0.$$

On the left-hand side, since $a_j^{n_\gamma} \to_\gamma a_j$ and $z_j^{n_\gamma} \to_\gamma z_j$ by Eq. (30), the part $\boldsymbol{t}_j \in (a_j, z_j)$ in the conditioned event can be replaced by $\boldsymbol{t}_j \in (a_j^{n_\gamma}, z_j^{n_\gamma})$, and the entire conditioned event can be replaced by the pivotal event $\Omega_i^{n_\gamma}$ by Eqs. (40) and (45). Since z_i is a limit point of $(t_i^n)_{n=1}^\infty$, with $\mathcal{V}_{i,z_i}$ continuous and $\mathcal{V}_{i,z_i}(z_i) = z_i$, the indicator function $\mathbf{1}\left[\boldsymbol{t}_j < z_i\right]$ can be replaced by $\mathbf{1}\left[\boldsymbol{t}_j < \mathcal{V}_{i,z_i^{n_\gamma}}(t_i^{n_\gamma})\right]$. Hence we obtain Ineq. (48), and the lemma is proved.

E.9 Proof of Corollary 1

Suppose not, say $b_* > r$ is a consequential atom of β_j^*. By Theorem 2, no bidder other than j has an atom at b_*. Let $(\delta_n)_{n=1}^\infty$ and $((a_j, z_j), (a_j^n, z_j^n)_{n=1}^\infty)$ be those specified in Lemma 7. For any $i \neq j$, define c_i^n by Eq. (39). Since $b_* > r$, $\lim_{n\to\infty}(b_* - \delta_n - \max_{i\neq j} c_i^n) = 0$; otherwise, since the auction is first-price, bidder j with types in (a_j^n, z_j^n) would deviate to a bid $d_j^n \in B_j^{m_n}$ such that $\max_{i\neq j} c_i^n < d_j^n < b_* - \delta_n$. With $I \setminus \{j\}$ finite, there exist an $i \in I \setminus \{j\}$ and an infinite subsequence $(n_k)_{k=1}^\infty$ along which $c_i^{n_k} = \max_{j'\in J\setminus\{j\}} c_{j'}^{n_k}$ for all k. As no one but j has an atom at b_*, Eq. (40) holds. Since $\lim_{n\to\infty}(b_* - \delta_n - \max_{i\neq j} c_i^n) = 0$, $c_i^{n_k} < b_* + \delta_{n_k}$ and $b_* + \delta_{n_k} - c_i^{n_k} = O(\delta_{n_k})$. Furthermore, b_* is consequential by hypothesis of the lemma. The rest of the proof is identical to the proof of Theorem 2 starting from Eq. (40).

F Details of the Payoff-Security Theorem

F.1 Proof of Lemma 13

With atom r, let the subsequences $(\bar\beta^n)_{n=1}^\infty$, $(\delta_n)_{n=1}^\infty$ and $((a_i, z_i), (a_i^n, z_i^n)_{n=1}^\infty)_{i\in I}$ be the ones specified in Lemma 7. By Corollary 1, r is an atom only for bidder j. Hence $\inf\{t_j : \beta_j^*(t_j) = r\} = a_j < z_j$ and $a_i = z_i$ for all $i \neq j$. Suppose, to the contrary of the lemma, that $a_j < r$.

By Eq. (28), the $\bar{\beta}^n$-bids from bidder j with types converging to a_j^n from above belong to $(r-\delta_n, r+\delta_n)$. Revealed preference from the viewpoint of these types, coupled with the consequentiality of r, implies that j's expected payoff from winning given type a_j^n is greater than or equal to $r-\delta_n$ (Lemma 6). Since r is not an atom of β^*_{-j}, this inequality is preserved at the limit. Hence $\overline{W}_j(r, a_j, \beta^*) \geq r$. As $\overline{W}_j(r, a_j, \beta^*)$ is a convex combination between j's use value a_j and the resale revenue extracted from other bidders, this inequality coupled with $a_j < r$ implies that there exists a $k \neq j$ for whom $z_k > r$. Pick $i \in I \setminus \{j\}$ such that

$$z_i = \max_{k \neq j} z_k.$$

Let $\Delta U_i^n(t_i) := U_i(b_i^n, t, \bar{\beta}^n) - U_i(l, t, \bar{\beta}^n)$ denote the payoff difference for i rendered by the bid increase from the losing bid l to

$$b_i^n := \min\left\{b_i \in B_i^{m_n} : b_i \geq r + \delta_n\right\}$$

at $\bar{\beta}^n$. Since r is not an atom of β_k^* unless $k = j$,

$$\lim_{n\to\infty} \Pr\left\{l < \max_{k\neq j} \bar{\beta}_k^n(\boldsymbol{t}_k) < b_i^n\right\} = 0. \tag{105}$$

We shall prove that $\limsup_{n\to\infty} \Delta U_i^n(z_i^n) > 0$. If true, that means for all large n, $\Delta U_i^n(z_i^n) > 0$ and hence, by continuity (Lemma 5), bidder i's types converging to z_i^n from below, which are supposed to bid l, would rather bid b_i^n, a desired contradiction.

To this end, use Eq. (10) to obtain

$$\begin{aligned}
\lim_{n\to\infty} \Delta U_i^n(z_i^n) &= \lim_{n\to\infty} \mathbb{E}\left[\mathbf{1}\left[b_i^n > \max_{k\neq i} \bar{\beta}_k^n(\boldsymbol{t}_k)\right]\left(W_i(\boldsymbol{t}_{-i} \mid b_i^n, z_i^n, \bar{\beta}^n) - b_i^n - L_i(\boldsymbol{t}_{-i} \mid z_i^n, \bar{\beta}^n)\right)\right] \\
&\geq \lim_{n\to\infty} \mathbb{E}\left[\mathbf{1}\left[b_i^n > \max_{k\neq i} \bar{\beta}_k^n(\boldsymbol{t}_k)\right]\left(z_i^n - r - L_{ij}(\boldsymbol{t}_{-i} \mid z_i^n, \bar{\beta}^n)\right)\right],
\end{aligned}$$

where the inequality uses the facts $\overline{W}_i(b_i^n, z_i^n, \bar{\beta}^n) \geq z_i^n$ to replace $\overline{W}_i$ with z_i^n, $b_i^n \to_n r$ to replace b_i^n with r, and Eqs. (31) and (105) to replace L_i with L_{ij}. With r consequential,

$$\lim_{n\to\infty} \Pr\left\{b_i^n > \max_{k\neq i} \bar{\beta}_k^n(\boldsymbol{t}_k)\right\} \geq \prod_{k\neq i} F_k(z_k) > 0.$$

Thus, to prove $\limsup_n \Delta U_i^n(z_i^n) > 0$ it suffices to show

$$\lim_{n\to\infty} \mathbb{E}\left[z_i^n - r - L_{ij}(\boldsymbol{t}_{-i} \mid z_i^n, \bar{\beta}^n) \,\middle|\, b_i^n > \max_{k\neq i} \bar{\beta}_k^n(\boldsymbol{t}_k)\right] > 0. \tag{106}$$

To show that, apply Lemma 22 to the case where $t_i^n = z_i^n$ and $c_i^n = l$ for each n. (The lemma is applicable due to the choice of i and Eq. (105), which implies Eq. (40) that is required for the lemma.) Thus, by Ineq. (90),

$$\lim_{n\to\infty} \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid z_i^n, \bar{\beta}^n) \middle| b_i^n > \max_{k\neq i} \bar{\beta}_k^n(\boldsymbol{t}_k); \boldsymbol{t}_j < z_i^n\right] < z_i - r.$$

Thus, with $L_i \geq 0$ by definition of the resale mechanism, $z_i - r > 0$ and hence $\Pr\{\boldsymbol{t}_j < z_i^n \mid \boldsymbol{t}_j < z_j^n\} > 0$ for all sufficiently large n. Therefore, since $L_i(t_{-i} \mid z_i^n, \bar{\beta}^n) > 0$ only if $t_j < z_i^n$,

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid z_i^n, \bar{\beta}^n) \middle| b_i^n > \max_{k\neq i} \bar{\beta}_k^n(\boldsymbol{t}_k)\right] &\leq \lim_{n\to\infty} \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid z_i^n, \bar{\beta}^n) \middle| b_i^n > \max_{k\neq i} \bar{\beta}_k^n(\boldsymbol{t}_k); \boldsymbol{t}_j < z_i^n\right] \\ &< z_i - r,\end{aligned}$$

which implies (106), as desired.

F.2 Proof of Lemma 15

Let a serious bid b_* be an inconsequential atom of β^*. By definition of inconsequentiality,, there exists a bidder i for whom b_* is not an atom of β_i^*, so that $\Pr\{\beta_k^*(\boldsymbol{t}_k) > b_*\} = 1$ and hence $\lim_{m\to\infty} \Pr\{\beta_k^m(\boldsymbol{t}_k) > b_*\} = 1$. Since $\cup_{m=1}^\infty B_i^m$ is dense in the space of serious bids due to Eq. (21), there is a sequence $(b_i^m)_{m=1}^\infty$ converging to b_* with $b_i^m \in B_i^m$ for each m. Then

$$\Pr\{\beta_i^*(\boldsymbol{t}_i) > b_*\} = \lim_{m\to\infty} \Pr\{\beta_i^m(\boldsymbol{t}_i) > b_i^m\} = 1. \tag{107}$$

To prove by contradiction, suppose b_* is an atom of β_k^* for all $k \neq i$. Denote for any m

$$\begin{aligned}\underline{b}_i^m &:= \min\{b \in B_i^m : \beta_i^m = b \text{ on some } (x, x') \subseteq T_i \text{ with } x < x'\}, \\ \underline{b}_i^* &:= \inf\{\beta_i^*(t_i') : t_i' > 0\}.\end{aligned}$$

For each $k \neq i$ and any m, denote

$$\begin{aligned}z_k^m &:= (\beta^m)_{k,\sup}^{-1}(\underline{b}_i^m), \\ z_k &:= (\beta^*)_{k,\sup}^{-1}(\underline{b}_i^*).\end{aligned}$$

Since $\beta^m \to \beta^*$, $\underline{b}_i^m \to_m \underline{b}_i^*$ and z_k is a limit point of $(z_k^m)_{m=1}^\infty$. Extracting a converging subsequence and relabeling if necessary, assume that $z_k^m \to_m z_k$ for each $k \neq i$. Note that $z_k > 0$ for each $k \neq i$, as b_* is an atom of β_k^*.

Note that $\underline{b}_i^*$ is not an atom of β_i^*. Otherwise, since $\underline{b}_i^* \geq b_*$ (Eq. (107)) and b_* is an atom of β_{-i}^*, $\underline{b}_i^*$ would be a consequential atom and hence by Corollary 1 and the fact $b_* \geq r$ we have $\underline{b}_i^* = b_* = r$, meaning that r is a consequential tie, contradicting Theorem 2. By the same token, for any $k \neq i$, the interval $(\underline{b}_i^*, \infty)$ contains no atom of β_k^*.

Choose a $j \neq i$ such that $z_j = \max_{k \neq i} z_k$. To derive a desired contradiction, we shall prove that some types of bidder j that are supposed to bid below $\underline{b}_i^m$ at the m-equilibrium would rather deviate to a slightly higher bid for large m.

Step 1: The price markup By definition of $\underline{b}_i^m$ and monotonicity of β_i^m, there is a sequence $(t_i^m)_{m=1}^\infty$ such that $\beta_i^m(t_i^m) = \underline{b}_i^m$ for each m and $t_i^m \to_m 0$. Then for all sufficiently large m, $z_j^m > t_i^m$.

Since $z_j = \max_{k \neq i} z_k$ and $z_k^m \to_m z_k$ for all $k \neq i$, for each m there exists $\epsilon_m \geq 0$ such that $z_k^m - \epsilon_m \leq z_j^m$ for each $k \notin \{i, j\}$ and $\epsilon_m \to 0$. Thus, for each $t_{-(i,j)} \in T_{-(i,j)}$ such that $t_k \leq z_k^m - \epsilon_m$ for each $k \notin \{i, j\}$, if bidder i wins with bid $\underline{b}_i^m$ (hence $t_k \leq z_k^m$ for all $k \neq i$), we have $z_j^m > t_i^m$ (for all large m) and

$$V_{k,\underline{b}_i^m,\beta^m}(t_k) \leq t_k \leq z_k^m - \epsilon_m \leq z_j^m \overset{(3)}{=} \max V_{j,\underline{b}_i^m,\beta^m},$$

hence the resale price $p_{j,i,\underline{b}_i^m,\beta^m}\left(t_i^m, t_{-(i,j)}\right)$ for j obeys Eq. (6). With the notation in (44),

$$p_{j,i,\underline{b}_i^m,\beta^m}\left(t_i^m, \boldsymbol{t}_{-(i,j)}\right) = \mathcal{V}_{j,z_j^m}^{-1}\left(\max\left\{t_i^m, \max_{k \notin \{i,j\}} \mathcal{V}_{k,z_k^m}(\boldsymbol{t}_k)\right\}\right).$$

Since $z_k^m \to_m z_k$, $\epsilon_m \to 0$, $t_i^m \to_m 0$, and the functions $\mathcal{V}_{k,z_k^m}$ and $x \mapsto \mathcal{V}_{k,x}^{-1}(v)$ are continuous,

$$\begin{aligned} &\lim_{m\to\infty} \mathbb{E}\left[p_{j,i,\underline{b}_i^m,\beta^m}\left(t_i^m, \boldsymbol{t}_{-(i,j)}\right) \mid \forall k \notin \{i,j\} : \boldsymbol{t}_k \leq z_k^m - \epsilon_m\right] \\ = \;& \mathbb{E}\left[\mathcal{V}_{j,z_j}^{-1}\left(\max\left\{0, \max_{k \notin \{i,j\}} \mathcal{V}_{k,z_k}(\boldsymbol{t}_k)\right\}\right) \middle| \forall k \notin \{i,j\} : \boldsymbol{t}_k \leq z_k\right]. \end{aligned}$$

By Eq. (52) and the fact $\mathcal{V}_{k,z_k^m} \to_m \mathcal{V}_{k,z_k}$ for each $k \neq i$,

$$\lim_{m\to\infty} \overline{W}_i\left(\underline{b}_i^m, t_i^m, \beta^m\right) = \mathbb{E}\left[\max\left\{0, \max_{k \neq i} \mathcal{V}_{k,z_k}(\boldsymbol{t}_k)\right\} \middle| \forall k \neq i : \boldsymbol{t}_k \leq z_k\right].$$

These two equations combined with Ineq. (53), which is due to $z_j = \max_{k \neq i} z_k > 0$, imply

$$\lim_{m\to\infty}\left(\mathbb{E}\left[p_{j,i,\underline{b}_i^m,\beta^m}\left(t_i^m, \boldsymbol{t}_{-(i,j)}\right) \mid \forall k \notin \{i,j\} : \boldsymbol{t}_k \leq z_k^m - \epsilon_m\right] - \overline{W}_i\left(\underline{b}_i^m, t_i^m, \beta^m\right)\right) > 0.$$

Since $\beta_i^m(t_i^m) = \underline{b}_i^m$, $\overline{W}_i\left(\underline{b}_i^m, t_i^m, \beta^m\right) \geq \underline{b}_i^m$ by Lemma 6, the above inequality implies

$$\lim_{m\to\infty}\left(\mathbb{E}\left[p_{j,i,\underline{b}_i^m,\beta^m}\left(t_i^m, \boldsymbol{t}_{-(i,j)}\right) \mid \forall k \notin \{i,j\} : \boldsymbol{t}_k \leq z_k^m - \epsilon_m\right] - \underline{b}_i^m\right) > 0.$$

Thus, there exists $\eta \in (0, z_j/\lambda)$ such that

$$\lim_{m\to\infty} \left(\mathbb{E}\left[p_{j,i,\underline{b}_i^m,\beta^m}\left(t_i^m, \boldsymbol{t}_{-(i,j)}\right) \mid \forall k \notin \{i,j\} : \boldsymbol{t}_k \leq z_k^m - \epsilon_m\right] - \underline{b}_i^m\right) - 5\eta > 0, \tag{108}$$

with λ the positive constant specified in Lemma 2.a.ii.

Step 2: Construct a deviation for bidder j For any m, if $b', b \in B_i^m$ then for each $k \neq i$, neither bids are atom of β_k^m, by Eq. (19). Hence both b_i' and b_i satisfy Eq. (4). If in addition $b' > b$ then, by Eqs. (3) and (4),

$$V_{k,b,\beta^m}(t_k) - V_{k,b',\beta^m}(t_k) = \begin{cases} -\frac{F_k\left((\beta^m)^{-1}_{k,\sup}(b')\right) - F_k\left((\beta^m)^{-1}_{k,\sup}(b)\right)}{f_k(t_k)} & \text{if } t_k \leq (\beta^m)^{-1}_{k,\sup}(b) \\ V_{k,b',\beta^m}(t_k) - (\beta^m)^{-1}_{k,\sup}(b) & \text{if } (\beta^m)^{-1}_{k,\sup}(b) \leq t_k \leq (\beta^m)^{-1}_{k,\sup}(b') \\ (\beta^m)^{-1}_{k,\sup}(b') - (\beta^m)^{-1}_{k,\sup}(b) & \text{if } t_k \geq (\beta^m)^{-1}_{k,\sup}(b'). \end{cases}$$

Consequently, with $f_k > 0$ on the compact T_k for all k,

$$|V_{k,b,\beta^m}(t_k) - V_{k,b',\beta^m}(t_k)| = O\left(F_k\left((\beta^m)^{-1}_{k,\sup}(b')\right) - F_k\left((\beta^m)^{-1}_{k,\sup}(b)\right)\right).$$

Thus, there exists $\xi > 0$ such that, for any m and any $k \neq i$,

$$\Pr\{b < \beta_k^m(\boldsymbol{t}_k) < b'\} < \xi \Longrightarrow \|V_{k,b',\beta^m} - V_{k,b,\beta^m}\|_{\sup} < \eta\lambda/|I|. \tag{109}$$

Since the limit $\underline{b}_i^*$ of $(\underline{b}_i^m)_{m=1}^\infty$ is not an atom of the limit β_i^* of $(\beta_i^m)_{m=1}^\infty$, and $(\underline{b}_i^*, \infty)$ contains no atom of the limit β_{-i}^* of $\left(\beta_{-i}^m\right)_{m=1}^\infty$, by Lemma 20 there exists $\delta \in (0, \eta/2)$ for which

$$\begin{aligned} \lim_{m\to\infty} \left(\sup\{t_i : \beta_i^m(t_i) \leq \underline{b}_i^m + \delta\}\right) &< \eta\lambda, &(110)\\ \lim_{m\to\infty} \Pr\{\exists k \neq i : \underline{b}_i^m < \beta_k^m(\boldsymbol{t}_k) \leq \underline{b}_i^m + \delta\} &< \min\left\{\xi, \eta, \frac{\eta}{\max_{k\in I} \bar{t}_k}\right\}. &(111)\end{aligned}$$

For each m, let

$$\begin{aligned} b_j^m &:= \min\left\{b \in B_j^m : b > \underline{b}_i^* + \delta\right\}, \\ c_j^m &:= \max\left\{\beta_j^m(t_j) : t_j < z_j^m\right\}, \\ \Delta U_j^m(t_j) &:= U_i(b_j^m, t_j, \beta^m) - U_i(c_j^m, t_j, \beta^m). \end{aligned}$$

Step 3: Bidder j's strict incentive to deviate For each m, by definition of c_j^m, there is a nondegenerate interval (x^m, z_j^m) such that $\beta_j^m(t_j) = c_j^m$ for all $t_j \in (x^m, z_j^m)$. Also note $c_j^m \leq \beta_j^m(z_j^m) < \underline{b}_i^m$ by monotonicity of β_j^m, the definition of z_j^m, and Eq. (19). Thus, since

$\underline{b}_i^m \to_m \underline{b}_i^*$, $c_j^m \leq \underline{b}_i^* < b_j^m$ for all sufficiently large m. We shall derive the desired contradiction by proving that for sufficiently large m some elements of (x^m, z_j^m) strictly prefer to deviate from their m-equilibrium bid c_j^m to the bid b_j^m. By continuity of ΔU_j^m (Lemma 5), it suffices to show $\lim_{m\to\infty} \Delta U_j^m(z_j^m) > 0$.

Substep 3.a: The probability of winning Now that $c_j^m < \underline{b}_i^m$ for large m,

$$\lim_{m\to\infty} \Pr\left\{\beta_i^m(\boldsymbol{t}_i) > c_j^m\right\} \geq \lim_{m\to\infty} \Pr\left\{\beta_i^m(\boldsymbol{t}_i) \geq \underline{b}_i^m\right\} = 1,$$

with the equality due to the definition of $\underline{b}_i^m$. Hence

$$\lim_{m\to\infty} U_j(c_j^m, t_j, \beta^m) = \lim_{m\to\infty} \mathbb{E}L_j(\boldsymbol{t}_{-j} \mid t_j, \beta^m)$$

for any $t_j \in T_j$ by Eq. (10). Thus, again by Eq. (10),

$$\lim_{m\to\infty} \Delta U_j^m(z_j) = \lim_{m\to\infty} \Pr\left\{b_j^m > \max_{k\neq j} \beta_k^m(\boldsymbol{t}_k)\right\} \overline{\Pi}_j^m(z_j^m),$$

where

$$\overline{\Pi}_j^m(z_j^m) := \mathbb{E}\left[W_j(\boldsymbol{t}_{-j} \mid b_j^m, z_j^m, \beta^m) - b_j^m - L_i(\boldsymbol{t}_{-j} \mid z_j^m, \beta^m) \middle| b_j^m > \max_{k\neq j} \beta_k^m(\boldsymbol{t}_k)\right]. \tag{112}$$

By definition of $\underline{b}_i^*$ being $\inf\{\beta_i^*(t_i') : t_i' > 0\}$, $\Pr\left\{\underline{b}_i^* \leq \beta_i^*(\boldsymbol{t}_i) \leq \underline{b}_i^* + \delta\right\} > 0$. Consequently, by definition of b_j^m as well as the fact that $z_k > 0$ for all $k \neq i$,

$$\lim_{m\to\infty} \Pr\left\{b_j^m > \max_{k\neq j} \beta_k^m(\boldsymbol{t}_k)\right\} \geq \Pr\left\{\underline{b}_i^* \leq \beta_i^*(\boldsymbol{t}_i) \leq \underline{b}_i^* + \delta\right\} \prod_{k\notin\{i,j\}} F_k(z_k) > 0.$$

Thus, it suffices to show $\lim_{m\to\infty} \overline{\Pi}_j^m(z_j^m) > 0$.

Substep 3.b: The resale prices By (111), for any large enough m and any $k \neq i$,

$$\underline{b}_i^m \leq \beta_i^m(t_i) \leq \underline{b}_i^m + \delta \Rightarrow \Pr\left\{\underline{b}_i^m < \beta_k^m(\boldsymbol{t}_k) < \beta_i^m(t_i)\right\} < \xi \overset{(109)}{\Rightarrow} \left\| V_{k,\beta_i^m(t_i),\beta^m} - V_{k,\underline{b}_i^m,\beta^m} \right\|_{\sup} < \eta\lambda/|I|.$$

Thus, by (110) and the fact $t_i^m \to_m 0$, we have for any sufficiently large m, any $t_{-(i,j)} \in T_{-(i,j)}$ and any t_i such that $\underline{b}_i^m \leq \beta_i^m(t_i) \leq \underline{b}_i^m + \delta$,

$$\max\left\{t_i, \max_{k\notin\{i,j\}} V_{k,\beta_i^m(t_i),\beta^m}(t_k)\right\} - \max\left\{t_i^m, \max_{k\notin\{i,j\}} V_{k,\underline{b}_i^m,\beta^m}(t_k)\right\} > -2\eta\lambda. \tag{113}$$

By Eq. (6), the two terms on the left-hand side of (113) can be inverted into j's resale prices via the inverses of his posterior virtual utility functions. Thus, with the derivatives of the inverses bounded from above by $1/\lambda$ (Lemma 2.a.ii). Hence

$$p_{j,i,\beta_i^m(t_i),\beta^m}\left(t_i, t_{-(i,j)}\right) - p_{j,i,\underline{b}_i^m,\beta^m}\left(t_i^m, t_{-(i,j)}\right) > -2\eta \tag{114}$$

for any $t_{-(i,j)} \in T_{-(i,j)}$ such that $t_k \leq z_k^m - \epsilon_m$ for each $k \notin \{i,j\}$ and any t_i such that $\underline{b}_i^m \leq \beta_i^m(t_i) \leq \underline{b}_i^m + \delta$. The applicability of Eq. (6) to the resale price $p_{j,i,\underline{b}_i^m,\beta^m}\left(t_i^m, t_{-(i,j)}\right)$ has been explained at Step 1, and its applicability to $p_{j,i,\beta_i^m(t_i),\beta^m}\left(t_i, t_{-(i,j)}\right)$, with $\beta_i^m(t_i)$ playing the role b_i, is because for large m we have $t_i < z_j^m$ ($\underline{b}_i^m \leq \beta_i^m(t_i) \leq \underline{b}_i^m + \delta$ implies via (110) that $t_i < \eta\lambda < z_i^m$ for large m) and, for each $k \notin \{i,j\}$,

$$V_{k,\beta_i^m(t_i),\beta^m}(t_k) \leq t_k \leq z_k^m - \epsilon_m \leq z_j^m = (\beta^m)_{j,\sup}^{-1}\left(\underline{b}_i^m\right) \leq (\beta^m)_{j,\sup}^{-1}\left(\beta_i^m(t_i)\right).$$

Integrating (114) across the $(t_i, t_{-(i,j)})$ quantified above, we have, for all large m,

$$\begin{aligned}
& \mathbb{E}\left[p_{j,i,\beta_i^m(\boldsymbol{t}_i),\beta^m}\left(\boldsymbol{t}_i, \boldsymbol{t}_{-(i,j)}\right) \mid \underline{b}_i^m \leq \beta_i^m(\boldsymbol{t}_i) < b_j^m; \forall k \notin \{i,j\}\left[\boldsymbol{t}_k \leq z_k^m - \epsilon_m\right]\right] \\
\geq \; & \mathbb{E}\left[p_{j,i,\underline{b}_i^m,\beta^m}\left(t_i^m, \boldsymbol{t}_{-(i,j)}\right) \mid \forall k \notin \{i,j\}\left[\boldsymbol{t}_k \leq z_k^m - \epsilon_m\right]\right] - 2\eta.
\end{aligned}$$

This combined with Ineq. (108) and $\delta < \eta/2$ (which implies $b_j^m < \underline{b}_i^* + \eta$ by the definition of b_j^m; then by $\underline{b}_i^m \to_m \underline{b}_i^*$ we have $b_j^m < \underline{b}_i^m + \eta$ for all large m) gives

$$\begin{aligned}
& \lim_{m\to\infty}\left(\mathbb{E}\left[p_{j,i,\beta_i^m(\boldsymbol{t}_i),\beta^m}\left(\boldsymbol{t}_i, \boldsymbol{t}_{-(i,j)}\right) \mid \underline{b}_i^m \leq \beta_i^m(\boldsymbol{t}_i) < b_j^m; \forall k \notin \{i,j\}\left[\boldsymbol{t}_k \leq z_k^m - \epsilon_m\right]\right] - b_j^m\right) \\
\geq \; & \lim_{m\to\infty}\left(\mathbb{E}\left[p_{j,i,\underline{b}_i^m,\beta^m}\left(t_i^m, \boldsymbol{t}_{-(i,j)}\right) \mid \forall k \notin \{i,j\}\left[\boldsymbol{t}_k \leq z_k^m - \epsilon_m\right]\right] - 2\eta - \left(\underline{b}_i^m + \eta\right)\right) \\
> \; & 5\eta - 2\eta - \eta \; = \; 2\eta.
\end{aligned} \tag{115}$$

Substep 3.c: Bidder j's opportunity cost of winning:

$$\begin{aligned}
& \lim_{m\to\infty} \mathbb{E}\left[L_{ji}(\boldsymbol{t}_{-j} \mid z_j^m, \beta^m) \,\middle|\, b_j^m > \max_{k\neq j} \beta_k^m(\boldsymbol{t}_k)\right] \\
\overset{(111)}{\leq} \; & \lim_{m\to\infty} \mathbb{E}\left[L_{ji}(\boldsymbol{t}_{-j} \mid z_j^m, \beta^m) \,\middle|\, b_j^m > \max_{k\neq j} \beta_k^m(\boldsymbol{t}_k); \underline{b}_i^m > \max_{k\notin\{i,j\}} \beta_k^m(\boldsymbol{t}_k)\right] + \eta \\
= \; & \lim_{m\to\infty} \mathbb{E}\left[L_{ji}(\boldsymbol{t}_{-j} \mid z_j^m, \beta^m) \,\middle|\, b_j^m > \max_{k\neq j} \beta_k^m(\boldsymbol{t}_k); \forall k \notin \{i,j\}\left[\boldsymbol{t}_k \leq z_k^m - \epsilon_m\right]\right] + \eta \\
= \; & \lim_{m\to\infty} \mathbb{E}\left[z_j^m - p_{j,i,\beta_i^m(\boldsymbol{t}_i),\beta^m}\left(\boldsymbol{t}_i, \boldsymbol{t}_{-(i,j)}\right) \mid b_j^m > \beta_i^m(\boldsymbol{t}_i); \forall k \notin \{i,j\}\left[\boldsymbol{t}_k \leq z_k^m - \epsilon_m\right]\right] + \eta \\
\overset{(115)}{<} \; & \lim_{m\to\infty}\left(z_j^m - b_j^m\right) - \eta;
\end{aligned}$$

here the first equality is because the difference between the events $\underline{b}_i^m > \max_{k\notin\{i,j\}} \beta_k^m(\boldsymbol{t}_k)$ and $\forall k \notin \{i,j\}\, [\boldsymbol{t}_k \le z_k^m - \epsilon_m]$ vanishes as $m \to \infty$; the second equality is due to $t_k \le z_k^m - \epsilon_m \le z_j^m$ for all $k \notin \{i,j\}$ and $t_i < z_j^m$ for large m (due to Ineq. (110) and $\eta < z_j$). Ineq. (111) also implies that for any $k \notin \{i,j\}$

$$\lim_{m\to\infty} \mathbb{E}\left[\mathbf{1}\left[\beta_k^m(\boldsymbol{t}_k) > \max_{l\notin\{j,k\}} \beta_l^m(\boldsymbol{t}_l)\right] L_{jk}(\boldsymbol{t}_{-j} \mid z_j^m, \beta^m) \,\middle|\, b_j^m > \max_{k\neq j} \beta_k^m(\boldsymbol{t}_k)\right] < \eta.$$

Combining the two inequalities displayed above with Eq. (9), we have

$$\lim_{m\to\infty} \mathbb{E}\left[L_j(\boldsymbol{t}_{-j} \mid z_j^m, \beta^m) \,\middle|\, b_j^m > \max_{k\neq j} \beta_k^m(\boldsymbol{t}_k)\right] < \lim_{m\to\infty} \left(z_j^m - b_j^m\right).$$

Therefore, by Eq. (112) and the fact $W_j(\boldsymbol{t}_{-j} \mid b_j^m, z_j^m, \beta^m) \ge z_j^m$,

$$\lim_{m\to\infty} \overline{\Pi}_j^m(z_j^m) \ge \lim_{m\to\infty} \left(z_j^m - b_j^m - \mathbb{E}\left[L_j(\boldsymbol{t}_{-j} \mid z_j^m, \beta^m) \,\middle|\, b_j^m > \max_{k\neq j} \beta_k^m(\boldsymbol{t}_k)\right]\right) > 0,$$

as desired.

G Proof of Lemma 16

We start with two lemmas, given the hypothesis that a sequence $(b_i^m)_{m=1}^\infty$ of grid points converges to a b_i that is not an atom of β_{-i}^*. By Eq. (19), none of b_i^m is an atom of β_{-i}^m.

Lemma 23 *For any $k \in I$ and any $t_k \in \left[0, (\beta^*)_{k,\sup}^{-1}(b_i)\right)$,*

$$\lim_{m\to\infty} V_k(t_k \mid b_i^m, \beta^m) = V_k(t_k \mid b_i, \beta^*). \tag{116}$$

Proof Since neither b_i^m is an atom of β_k^m nor b_i an atom of β_k^*, Lemma 2.a.i is applicable. Thus, for any $t_k \in \left[0, (\beta^*)_{k,\sup}^{-1}(b_i)\right)$, $V_k(t_k \mid b_i, \beta^*)$ obeys Eq. (4) with $(\beta^*)_{k,\sup}^{-1}(b_i)$ being the $\beta_{k,\sup}^{-1}(b_i)$ there; for large enough m, such t_k also belongs to $\left[0, (\beta^m)_{k,\sup}^{-1}(b_i^m)\right]$ and hence $V_k(t_k \mid b_i^m, \beta^m)$ also obeys Eq. (4) with $(\beta^m)_{k,\sup}^{-1}(b_i^m)$ being the $\beta_{k,\sup}^{-1}(b_i)$ there. Since b_i is not an atom of β_k^*, the mass of $\boldsymbol{t}_k$ between $(\beta^m)_{k,\sup}^{-1}(b_i^m)$ and $(\beta^*)_{k,\sup}^{-1}(b_i)$ vanishes as $b_i^m \to b_i$. Thus, $F_k\left((\beta^m)_{k,\sup}^{-1}(b_i^m)\right) \to_m F_k\left((\beta^*)_{k,\sup}^{-1}(b_i)\right)$. Hence Eq. (116) follows. ■

Lemma 24 *For any $i \in I$, any $t_i \in T_i$ and any measurable subset $S \subseteq T_{-i}$,*

$$\lim_{m\to\infty} \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid t_i, \beta^m)\mathbf{1}[S]\right] = \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid t_i, \beta^*)\mathbf{1}[S]\right]. \tag{117}$$

Proof By definition of L_i in Eq. (9), $L_i(t_{-i} \mid t_i, \beta^*)$ equals zero unless t_{-i} belongs to the set

$$S' := \left\{t_{-i} \in T_{-i} : \beta^*_j(t_j) > l \text{ for some } j \neq i\right\}.$$

Since $\beta^m \to \beta^*$ and l is isolated from $[0, \infty)$, $\lim_{m\to\infty} L_i(t_{-i} \mid t_i, \beta^m) = 0$ by Eq. (9) unless $t_{-i} \in S'$. Thus, it suffices to prove (117) with the integration domain S replaced with $S \cap S'$.

First, consider any $t_{-i} \in S'$ at which the highest bid $\beta^*_j(t_j)$ among rivals of i is not an atom of β^*_{-j}. Then Eq. (116) holds for all $k \neq j$ with the role (b_i^m, b_i) played by $(\beta^m_j(t_j), \beta^*_j(t_j))$, and Eqs. (6) and (8) hold with respect to β^*. Eqs. (6) and (8) also hold with respect to β^m due to Eq. (19). Thus,

$$\lim_{m\to\infty} L_i(t_{-i} \mid t_i, \beta^m) = L_i(t_{-i} \mid t_i, \beta^*). \tag{118}$$

Second, consider the other kind of t_{-i} in S', the elements of

$$S'' := \left\{ t_{-i} \in S' : \exists j \neq i \left[\beta^*_j(t_j) \geq \max_{k \notin \{i,j\}} \beta^*_k(t_k); \beta^*_j(t_j) \text{ is an atom of } \beta^*_{-j}\right]\right\}.$$

Since there are at most countably many atoms of β^*_{-j}, we can discard any $t_{-i} \in S''$ such that $\beta^*_j(t_j)$ is not an atom of β^*_j, as all such t_{-i} constitute only a zero-measure subset of T_{-i}. Thus, suppose that $\beta^*_j(t_j)$ is an atom of both β^*_j and β^*_{-j}. Then $\beta^*_j(t_j)$ would be a tie, which is impossible by Theorem 2, unless $\beta^*_j(t_j)$ is inconsequential. Now that $\beta^*_j(t_j)$ is inconsequential, Lemma 15 (applicable because $\beta^*_j(t_j) > l$, as $t_{-i} \in S'$) implies that there are at least two bidders whose bid functions in β^* do not have $\beta^*_j(t_j)$ as an atom. One of them is a bidder k different than the i in this lemma, and Eq. (107) implies $\Pr\left\{\beta^*_k(\boldsymbol{t}_k) > \beta^*_j(t_j)\right\} = 1$ and $\lim_{m\to\infty}\left\{\beta^m_k(\boldsymbol{t}_k) > \beta^m_j(t_j)\right\} = 1$. The first equation says that those $t_{-(i,j)}$ at which $\beta^*_j(t_j)$ wins against β^*_{-j} constitute a zero-measure set, and the second says that the measure of those $t_{-(i,j)}$ at which $\beta^m_j(t_j)$ wins against β^m_{-j} shrinks to zero as $m \to \infty$. Thus,

$$\lim_{m\to\infty} \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid t_i, \beta^m)\mathbf{1}\left[S''\right]\right] = 0 = \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid t_i, \beta^*)\mathbf{1}\left[S''\right]\right].$$

Eq. (117) is obtained by summing this equation with the integration of Eq. (118) across all $t_{-i} \in S \cap S' \setminus S''$. ∎

Now that b_i is not an atom of β^*_{-i}, nor b_i^m an atom of β^m_{-i}, the winning event is simplified by Eq. (22). Thus, by Eq. (10), with the symbol t_i suppressed,

$$\begin{aligned} U_i(b_i^m, \beta^m) &= \mathbb{E}\left[\mathbf{1}\left[b_i^m > \max_{k\neq i} \beta^m_k(\boldsymbol{t}_k)\right](W_i(\boldsymbol{t}_{-i} \mid b_i^m, \beta^m) - b_i^m - L_i(\boldsymbol{t}_{-i} \mid \beta^m))\right] + \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid \beta^m)\right], \\ U_i(b_i, \beta^*) &= \mathbb{E}\left[\mathbf{1}\left[b_i > \max_{k\neq i} \beta^*_k(\boldsymbol{t}_k)\right](W_i(\boldsymbol{t}_{-i} \mid b_i, \beta^*) - b_i - L_i(\boldsymbol{t}_{-i} \mid \beta^*))\right] + \mathbb{E}\left[L_i(\boldsymbol{t}_{-i} \mid \beta^*)\right]. \end{aligned}$$

Here $W_i(\boldsymbol{t}_{-i} \mid b_i^m, \beta^m)$ and $W_i(\boldsymbol{t}_{-i} \mid b_i, \beta^*)$ obey Eq. (7) with virtual utility functions $(V_k(\cdot \mid b_i^m, \beta^m))_{k\neq i}$ and $(V_k(\cdot \mid b_i, \beta^*))_{k\neq i}$, because the condition for (7) is guaranteed by the indicator functions $\mathbf{1}\left[b_i^m > \max_{k\neq i} \beta_k^m(\boldsymbol{t}_k)\right]$ and $\mathbf{1}\left[b_i > \max_{k\neq i} \beta_k^*(\boldsymbol{t}_k)\right]$. Since $b_i^m \to_m b_i$ and b_i is not an atom of β_{-i}^*, Eq. (116) applies. Thus, by Eqs. (6) and (7),

$$\lim_{m\to\infty} W_i(t_{-i} \mid b_i^m, \beta^m) = W_i(t_{-i} \mid b_i, \beta^*) \quad \text{a.e. } t_{-i} \in T_{-i}.$$

As $\beta^m \to \beta^*$ and b_i is not an atom of β_{-i}^*, we also have

$$\lim_{m\to\infty} \mathbf{1}\left[b_i^m > \max_{k\neq i} \beta_k^m(t_k)\right] = \mathbf{1}\left[b_i > \max_{k\neq i} \beta_k^*(t_k)\right] \quad \text{a.e. } t_{-i} \in T_{-i}.$$

Combining these two equations with Eq. (117) yields the conclusion of the lemma.

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